1. Give all possible representations of 2022 as a sum of at least two consecutive positive integers and prove that these are the only representations.

Let \( S_{M,N} = \sum_{n=M}^{N} n \) be the sum of consecutive positive integers from \( n = M \) to \( n = N \) and recall the formula
\[
S_{1,N} = \frac{N(N + 1)}{2}
\]

We aim to solve the equation \( S_{M,N} = 2022 \). Using the above formula, observe
\[
S_{M,N} = S_{1,N} - S_{1,M-1}
= \frac{N(N + 1)}{2} - \frac{(M - 1)M}{2}
= \frac{N^2 + (M + N) - M^2}{2}
\]

Hence, an equivalent formulation for our equation is
\[
\frac{N^2 + (M + N) - M^2}{2} = 4044 \implies (N + M)(N - M + 1) = 2^2 \cdot 3 \cdot 337
\]

\( N + M \) and \( N - M + 1 \) must have opposite parity. This leads to checking 6 possible cases, of which we can quickly eliminate the possibilities corresponding to \( N + M \in \{3, 4, 12\} \):

(a) **Case 1:** \( N + M = 337, \ N - M + 1 = 12 \)

The system of equations reduces to \( 2N = 348 \), or \( N = 174 \). Then we find \( M = 163 \). Thus, \( S_{163,174} = 2022 \).

(b) **Case 2:** \( N + M = 1011, \ N - M + 1 = 4 \)

The system of equations reduces to \( 2N = 1014 \), or \( N = 507 \). Then we find \( M = 504 \). Thus, \( S_{504,507} = 2022 \).

(c) **Case 3:** \( N + M = 1348, \ N - M + 1 = 3 \)

The system of equations reduces to \( 2N = 1350 \), or \( N = 675 \). Then we find \( M = 673 \). Thus, \( S_{673,675} = 2022 \).
2. Let $A$ and $B$ be the two foci of an ellipse and let $P$ be a point on this ellipse. Prove that the focal radii of $P$ (that is, the segments $AP$ and $BP$) form equal angles with the tangent to the ellipse at $P$.

Let $\ell$ be the line passing through $P$ such that $AP$ and $BP$ form equal angles with $\ell$. It suffices to show that the line $\ell$ is tangent to the ellipse.

If $\ell$ is not tangent to the ellipse, then it intersects the ellipse both at $P$ and at a point $Q$ different from $P$. Let $A'$ be the reflection of $A$ with respect to the line $\ell$. Since $AP$ and $BP$ form equal angles with $\ell$, the points $A'$, $P$, and $B$ are collinear. It follows that

$$|A'B| = |A'P| + |PB| = |AP| + |PB| = |AQ| + |QB| = |A'Q| + |QB|.$$

By the triangle inequality, this is impossible unless $A'$, $Q$, and $B$ are on the same line, i.e. $P = Q$, a contradiction.
3. Find all positive integers $a, b, c, d,$ and $n$ satisfying $n^a + n^b + n^c = n^d$ and prove that these are the only such solutions.

Without loss of generality, $a \leq b \leq c \leq d$. Then dividing through by $n^d$, we have

$$n^{a-d} + n^{b-d} + n^{c-d} = 1$$

Each of the exponents is at most $-1$, so we have

$$n^{a-d} + n^{b-d} + n^{c-d} \leq \frac{3}{n}$$

which forces $1 \leq n \leq 3$. Clearly $n \neq 1$.

For $n = 3$, we must have all exponents equal to $-1$; that is, $a - d = -1$, $b - d = -1$, and $c - d = -1$. Hence, for any natural number $k \geq 2$, it follows that

$$a = k - 1$$
$$b = k - 1$$
$$c = k - 1$$
$$d = k$$

gives a solution to the equation.

For $n = 2$, we can have neither $\min\{a - d, b - d, c - d\} \geq -1$ nor $\max\{a - d, b - d, c - d\} \leq -2$, hence $c - d = -1$. The equation then reduces to

$$n^{a-d} + n^{b-d} = \frac{1}{2}$$

and by a similar argument, we must have $a - d = -2$ and $b - d = -2$. Then, for any natural number $k \geq 3$, it follows that

$$a = k - 2$$
$$b = k - 2$$
$$c = k - 1$$
$$d = k$$
4. Calculate the exact value of the series \( \sum_{n=2}^{\infty} \log(n^3 + 1) - \log(n^3 - 1) \) and provide justification.

Using properties of logarithms, we have

\[
\sum_{n=2}^{\infty} \log(n^3 + 1) - \log(n^3 - 1) = \sum_{n=2}^{\infty} \log \left( \frac{n^3 + 1}{n^3 - 1} \right)
= \log \left( \prod_{n=2}^{\infty} \frac{n^3 + 1}{n^3 - 1} \right)
= \log \left( \prod_{n=2}^{\infty} \frac{(n + 1)(n^2 - n + 1)}{(n - 1)(n^2 + n + 1)} \right)
= \log \left( \lim_{N \to \infty} \prod_{n=2}^{N} \frac{(n + 1)(n^2 - n + 1)}{(n - 1)(n^2 + n + 1)} \right)
\]

Then the product satisfies

\[
\prod_{n=2}^{N} \frac{n + 1}{n - 1} = \frac{(3)(4)(5)(6)(7) \cdots (N + 1)}{(1)(2)(3)(4)(5) \cdots (N - 1)} = \frac{1}{2} \cdot \left( \frac{N + 1}{N - 1} \right)
\]

and

\[
\prod_{n=2}^{N} \frac{n^2 - n + 1}{n^2 + n + 1} = \frac{(3)(7)(13)(21)(31) \cdots (N^2 - N + 1)}{(7)(13)(21)(31) \cdots (N^2 + N + 1)} = 3 \cdot \left( \frac{N^2 - N + 1}{N^2 + N + 1} \right)
\]

Letting \( N \to \infty \), we have

\[
\sum_{n=2}^{\infty} \log(n^3 + 1) - \log(n^3 - 1) = \log \left( \frac{3}{2} \right)
\]
5. Let $A$ be an invertible $n \times n$ matrix with complex entries. Suppose that for each positive integer $m$, there exists a positive integer $k_m$ and an $n \times n$ invertible matrix $B_m$ such that $A^{k_m} = B_mAB_m^{-1}$. Show that all eigenvalues of $A$ are equal to 1.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$ (which are allowed to repeat). Then $\lambda_1^{k_m}, \ldots, \lambda_n^{k_m}$ are the eigenvalues of $A^{k_m}$, and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $B_mAB_m^{-1}$. Therefore, we have for all $m \in \mathbb{N}$ that

$$\{\lambda_1^{k_m}, \ldots, \lambda_n^{k_m}\} = \{\lambda_1, \ldots, \lambda_n\}.$$ 

So for each $1 \leq i \leq n$ and $m \in \mathbb{N}$, $\lambda_i^{k_m} = \lambda_{a_{m,i}}$ for some $a_{m,i} \in \{1, \ldots, n\}$. By the pigeonhole principle, there exist $m, m' \in \mathbb{N}, k_m \neq k_{m'}$ such that $\lambda_i^{k_m} = \lambda_i^{k_{m'}}$ and thus $\lambda_i^{k_m - k_{m'}} = 1$. Hence, $\lambda_i$ is a root of unity for all $1 \leq i \leq n$.

Then there exists $m \in \mathbb{N}$ such that $\lambda_1^m = \cdots = \lambda_n^m = 1$. Then $\{1\} = \{\lambda_1^{k_m}, \ldots, \lambda_n^{k_m}\} = \{\lambda_1, \ldots, \lambda_n\}$. We are done.
6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function whose second derivative is continuous. Suppose that \( f \) and \( f'' \) are bounded. Show that \( f' \) is also bounded.

Assume that \( |f(x)|, |f''(x)| \leq M \) for some \( M > 0 \) for all \( x \in \mathbb{R} \). If \( f' \) is not bounded, then there exists \( x_0 \in \mathbb{R} \) such that \( |f'(x_0)| > N \) for some \( N \in \mathbb{R} \) to be chosen later. We prove the case where \( f'(x_0) > N \); the proof of the other case is similar.

Since \( |f''(x)| \leq M \) for all \( x \in \mathbb{R} \), by the Mean Value Theorem, we have

\[
f'(x) > \frac{N}{2} \quad \text{for all } x \text{ such that } x_0 - \frac{N}{2M} \leq x \leq x_0
\]

since if not, then there exists \( y \in [x_0, x] \) such that

\[
f''(y) = \frac{f'(x_0) - f'(x)}{x - x_0} > \frac{N/2}{N/2M} = M,
\]
a contradiction. Define \( y_0 := x_0 - \frac{N}{2M} \). Again by the Mean Value Theorem,

\[
f(x_0) - f(y_0) \geq \frac{N^2}{4M}
\]
since if not, then there exists \( y_0 \leq z \leq x_0 \) such that

\[
f'(z) = \frac{f(x_0) - f(y_0)}{x_0 - y_0} < \frac{N^2/4M}{N/2M} = N/2,
\]
a contradiction. Hence one of \( |f(x_0)| \) and \( |f(y_0)| \) is at least \( N^2/8M \). We arrive at a contradiction if we choose \( N \) so that \( N^2 > 8M^2 \) since we then have

\[
\max\{|f(x_0)|, |f(y_0)|\} \geq \frac{N^2}{8M} > M
\]