40th VTRMC, 2018, Solutions

1. Let \( I = \int_1^2 \frac{\arctan(1+x)}{x} \, dx \). First we integrate by parts to obtain

\[
I = [\ln(x) \arctan(1+x)]_1^2 - \int_1^2 \frac{\ln x}{1+(1+x)^2} \, dx
\]

\[
= \ln(2) \arctan(3) - \int_1^2 \frac{\ln x}{2+2x+x^2} \, dx.
\]

Now let \( J = \int_1^2 \frac{\ln x}{2+2x+x^2} \, dx \) and make the substitution \( x = 2/y \). We obtain

\[
J = \int_2^1 \frac{\ln 2 - \ln y}{2+4/y+4/y^2} (-2/y^2) \, dy = \int_1^2 \frac{\ln 2}{1+(1+y)^2} \, dy - J.
\]

Therefore \( 2J = \int_1^2 \frac{\ln^2 2}{1+(1+y)^2} \, dy = [\ln(2) \arctan(1+y)]_1^2 = \ln(2)(\arctan(3) - \arctan(2)) \) and we deduce that \( I = \ln(2)(\arctan(3) + \arctan(2))/2 \). Now \( \tan(\arctan(3) + \arctan(2)) = (3+2)(1-6) = -1 \), which shows that \( \arctan(3) - \arctan(2) = 3\pi/4 \). Therefore \( I = 3\pi \ln(2)/8 \), and the answer is \( q = 3/8 \).

2. First we’ll show that if \( X, Y \in M_6(\mathbb{Z}) \), \( X \equiv I \equiv Y \mod 3 \), and \( XYX = Y \), then \( X = I \). Suppose \( X \neq I \) and write \( X = I + pC \) where \( p \) is a positive power of 3 and \( C \not\equiv 0 \mod 3 \). Note that \( XYX = Y^r \) for all odd integers \( r \). Write \( Y = I + 3D \) where \( D \in M_6(\mathbb{Z}) \). Then \( Y^p \equiv I \mod 3p \), so \( X^2 \equiv I \mod 3p \). Therefore \( I + 2pC + p^2C \equiv I \mod 3p \) which is not the case. Thus \( X = I \) and we conclude that \( A^3 = I \). Now write \( A = I + qD \) where \( q \) is a positive power of 3 and \( D \not\equiv 0 \mod 3 \). Then \( (I+qD)^3 \equiv I \mod 9q \), which shows that \( 3qD \equiv 0 \mod 9q \) which is not the case.

3. Let \( \mathbb{M} = \{2, 3, \ldots\} = \mathbb{N} \setminus \{1\} \). Then \( f^2(\mathbb{N}) = \mathbb{M} \) and therefore \( f(\mathbb{N}) = \mathbb{N} \) or \( \mathbb{M} \). The former yields \( f^2(\mathbb{N}) = \mathbb{N} \), which is not the case, so we must have the latter which yields \( f(\mathbb{M}) = \mathbb{M} \). It follows that \( f^2(\mathbb{M}) = \mathbb{M} \) and we have a contradiction, so there is no such \( f \), as required.

4. Let \( d = \gcd(m,n) \). Then \( d = an + bm \) for some integers \( a \) and \( b \). Now \( \binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} \), therefore

\[
\frac{d}{n} \binom{n}{m} = (a+bm/n) \binom{n}{m} = a \binom{n}{m} + b \binom{n-1}{m-1}.
\]

Since \( \binom{n}{m} \) and \( \binom{n-1}{m-1} \) are integers, the result follows.
5. We’ll show that \((a_n)\) is unbounded. We have \(a_{n-1} = \int_0^{1/\sqrt{n-1}} \frac{|1-e^{it}|}{|1-e^{it}|} dt\). Note that \(|1-e^{it}| \leq t\) for \(t \geq 0\). To see this, by squaring both sides, this is equivalent to \(2 - 2\cos t \leq t^2\), i.e. \(t^2 + 2\cos t - 2 \geq 0\), which is true because we have equality when \(t = 0\), and the derivative of the left hand side is non-negative for \(t \geq 0\) by using the inequality \(sint \leq t\) for \(t \geq 0\). Therefore it will be sufficient to show that \(b_n := \int_0^{1/\sqrt{n-1}} |1-e^{it}|/t \ dt\) is unbounded (because \(\pi/4 < 1\)). However for \(n \in \mathbb{Z}\),

\[
\int_{\pi r/n}^{\pi(r+1)/n} |1-e^{it}| \ dt = \int_{\pi r/n}^{\pi(r+1)/n} \sqrt{2-2\cos nt} = 4/n.
\]

Let \(k = [\sqrt{n-1}/\pi]\), so \(k\) is the greatest positive integer such that \(k\pi < \sqrt{n-1}\). Note that \(k \to \infty\) as \(n \to \infty\). Then \(b_n \geq 4/\pi (1 + 1/2 + \cdots + 1/k)\), which is unbounded because the harmonic series is divergent.

6. First we show that \(a_n - b_n \geq 0\) for all \(n \geq 1\). This is equivalent to proving

\[
(1 - \frac{1}{n})(1/2 + 1/4 + \cdots + \frac{1}{2n}) \leq 1 + 1/3 + \cdots + \frac{1}{2n-1},
\]

that is

\[
1 + 1/2 + 1/3 + \cdots + 1/n \leq n((2-1) + (2/3 - 2/4) + \cdots + (\frac{2}{2n-1} - \frac{2}{2n})).
\]

Since \(1 + 1/2 + \cdots + 1/n \leq n\), the assertion follows. Since \(a_1 - b_1 = 0\), we see that the minimum of \(a_n - b_n\) is zero.

Next we show that \(a_n - b_n\) is decreasing for \(n\) sufficiently large. We have

\[
(a_n - b_n) - (a_{n+1} - b_{n+1}) = a_n - a_{n+1} - (b_n - b_{n+1})
\]

\[
= \frac{1}{(n+1)(n+2)} \left(1 + 1/3 + \cdots + \frac{1}{2n-1}\right) - \frac{1}{(n+2)(2n+1)}
\]

\[
- \frac{1}{n(n+1)} \left(1/2 + 1/4 + \cdots + \frac{1}{2n}\right) + \frac{1}{(n+1)(2n+2)}
\]

Now \(\frac{1}{(n+1)(2n+2)} - \frac{1}{(n+2)(2n+1)} > 0\) for all \(n \geq 1\), so we need to prove

\[
\frac{1}{(n+1)(n+2)} \left(1 + 1/3 + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n(n+1)} \left(1/2 + 1/4 + \cdots + \frac{1}{2n}\right)
\]
for \( n \) sufficiently large. Multiplying by \( n(n + 1)(n + 2) \) and then subtracting \( n(1/2 + 1/4 + \cdots + \frac{1}{2n}) \) from both sides, means we want to prove

\[
n(1/2 + 1/12 + \cdots + \frac{1}{(2n - 1)2n}) > 1 + 1/2 + \cdots + 1/n
\]

for sufficiently large \( n \). However this is clear for \( n \geq 4 \). Therefore \( a_n - b_n \) takes its maximum value for some \( n \leq 4 \). By inspection, the maximum value occurs when \( n = 3 \), which is \( 7/90 \).

7. Note that if \( g \) and \( h \) are continuous piecewise-monotone functions on \([a, b]\), then \( \ell(gh) \leq \ell(g)\ell(h) \). Thus \( \ell(f^n) \leq (\ell(f))^n \) for all \( n \in \mathbb{N} \). Now fix a positive integer \( k \). Given \( n \in \mathbb{N} \), there are integers \( q \) and \( r \) such that \( n = qk + r \) with \( 0 \leq r < k \). Then \( \ell(f^n) \leq (\ell(f^k))^q(\ell(f))^r \), consequently

\[
\sqrt[n]{\ell(f^n)} \leq (\ell(f^k))^q/n(\ell(f))^r/n.
\]

Since \( k \) is fixed, \( r/n \to 0 \) and \( q/n \to 1/k \) as \( n \to \infty \). Therefore \( \limsup \sqrt[n]{\ell(f^n)} \leq \ell(f^k) \) and we deduce that

\[
\limsup \sqrt[n]{\ell(f^n)} \leq \inf \frac{1}{k} \ell(f^k) \leq \liminf \sqrt[n]{\ell(f^k)}
\]

and the result follows.