40th VTRMC, 2018, Solutions

1. Let $I = \int_1^2 \frac{\arctan(1+x)}{x}\,dx$. First we integrate by parts to obtain

$$I = [\ln(x)\arctan(1+x)]_1^2 - \int_1^2 \frac{\ln x}{1+(1+x)^2}\,dx$$
$$= \ln(2)\arctan(3) - \int_1^2 \frac{\ln x}{2+2x+x^2}\,dx.$$

Now let $J = \int_1^2 \frac{\ln x}{2+2x+x^2}\,dx$ and make the substitution $x = 2/y$. We obtain

$$J = \int_2^1 \frac{\ln 2 - \ln y}{2+4/y+4/y^2}(-2/y^2)\,dy = \int_1^2 \frac{\ln 2}{1+(1+y)^2}\,dy - J.$$

Therefore $2J = \int_1^2 \frac{\ln 2}{1+(1+y)^2}\,dy = [\ln(2)\arctan(1+y)]_1^2 = \ln(2)(\arctan(3) - \arctan(2))$ and we deduce that $I = \ln(2)(\arctan(3)+\arctan(2))/2$. Now $\tan(\arctan(3)+\arctan(2)) = (3+2)(1-6) = -1$, which shows that $\arctan(3) - \arctan(2) = 3\pi/4$. Therefore $I = 3\pi\ln(2)/8$, and the answer is $q = 3/8$.

2. First we'll show that if $X, Y \in \mathrm{M}_6(\mathbb{Z})$, $X \equiv I \equiv Y \bmod 3$, and $XYX = Y$, then $X = I$. Suppose $X \neq I$ and write $X = I + pC$ where p is a positive power of 3 and $C \not\equiv 0 \bmod 3$. Note that $XY^rX = Y^r$ for all odd integers r. Write $Y = I + 3D$ where $D \in \mathrm{M}_6(\mathbb{Z})$. Then $Y^p \equiv I \bmod 3p$, so $X^2 \equiv I \bmod 3p$. Therefore $I + 2pC + p^2C \equiv I \bmod 3p$ which is not the case. Thus $X = I$ and we conclude that $A^3 = I$. Now write $A = I + qD$ where q is a positive power of 3 and $D \not\equiv 0 \bmod 3$. Then $(I+qD)^3 \equiv I \bmod 9q$, which shows that $3qD \equiv 0 \bmod 9q$ which is not the case.

3. Let $\mathbb{M} = \{2,3,\dots\} = \mathbb{N} \setminus \{1\}$. Then $f^2(\mathbb{N}) = \mathbb{M}$ and therefore $f(\mathbb{N}) = \mathbb{N}$ or $\mathbb{M}$. The former yields $f^2(\mathbb{N}) = \mathbb{N}$, which is not the case, so we must have the latter which yields $f(\mathbb{M}) = \mathbb{M}$. It follows that $f^2(\mathbb{M}) = \mathbb{M}$ and we have a contradiction, so there is no such f, as required.

4. Let $d = \gcd(m,n)$. Then $d = an + bm$ for some integers a and b. Now $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$, therefore

$$\frac{d}{n}\binom{n}{m} = (a + bm/n)\binom{n}{m} = a\binom{n}{m} + b\binom{n-1}{m-1}.$$

Since $\binom{n}{m}$ and $\binom{n-1}{m-1}$ are integers, the result follows.

5. We'll show that (a_n) is unbounded. We have $a_{n-1} = \int_0^{1/\sqrt{n-1}} \frac{|1-e^{nit}|}{|1-e^{it}|}\,dt$. Note that $|1-e^{it}| \le t$ for $t \ge 0$. To see this, by squaring both sides, this is equivalent to $2-2\cos t \le t^2$, i.e. $t^2+2\cos t-2 \ge 0$, which is true because we have equality when $t=0$, and the derivative of the left hand side is non-negative for $t \ge 0$ by using the inequality $\sin t \le t$ for $t \ge 0$. Therefore it will be sufficient to show that $b_n := \int_0^{1/\sqrt{n-1}} |1-e^{nit}|/t\,dt$ is unbounded (because $\pi/4 < 1$). However for $n \in \mathbb{Z}$,

$$\int_{\pi r/n}^{\pi(r+1)/n} |1-e^{nit}|\,dt = \int_{\pi r/n}^{\pi(r+1)/n} \sqrt{2-2\cos nt} = 4/n.$$

Let $k = [\sqrt{n-1}/\pi]$, so k is the greatest positive integer such that $k\pi < \sqrt{n-1}$. Note that $k \to \infty$ as $n \to \infty$. Then $b_n \ge \frac{4}{\pi}(1+1/2+\cdots+1/k)$, which is unbounded because the harmonic series is divergent.

6. First we show that $a_n - b_n \ge 0$ for all $n \ge 1$. This is equivalent to proving

$$(1+\frac{1}{n})(1/2+1/4+\cdots+\frac{1}{2n}) \le 1+1/3+\cdots+\frac{1}{2n-1},$$

that is

$$1+1/2+1/3+\cdots+1/n \le n\big((2-1)+(2/3-2/4)+\cdots+(\frac{2}{2n-1}-\frac{2}{2n})\big).$$

Since $1+1/2+\cdots+1/n \le n$, the assertion follows. Since $a_1-b_1=0$, we see that the minimum of a_n-b_n is zero.

Next we show that a_n-b_n is decreasing for n sufficiently large. We have

$$\begin{aligned}(a_n-b_n)-(a_{n+1}-b_{n+1}) &= a_n-a_{n+1}-(b_n-b_{n+1})\\ &= \frac{1}{(n+1)(n+2)}(1+1/3+\cdots+\frac{1}{2n-1}) - \frac{1}{(n+2)(2n+1)}\\ &\quad - \frac{1}{n(n+1)}(1/2+1/4+\cdots+\frac{1}{2n}) + \frac{1}{(n+1)(2n+2)}.\end{aligned}$$

Now $\frac{1}{(n+1)(2n+2)} - \frac{1}{(n+2)(2n+1)} > 0$ for all $n \ge 1$, so we need to prove

$$\frac{1}{(n+1)(n+2)}(1+1/3+\cdots+\frac{1}{2n-1}) > \frac{1}{n(n+1)}(1/2+1/4+\cdots+\frac{1}{2n})$$

for n sufficiently large. Multiplying by $n(n+1)(n+2)$ and then subtracting $n(1/2+1/4+\cdots+\frac{1}{2n})$ from both sides, means we want to prove

$$n(1/2+1/12+\cdots+\frac{1}{(2n-1)2n}) > 1+1/2+\cdots+1/n$$

for sufficiently large n. However this is clear for $n \geq 4$. Therefore $a_n - b_n$ takes its maximum value for some $n \leq 4$. By inspection, the maximum value occurs when $n = 3$, which is 7/90.

7. Note that if g and h are continuous piecewise-monotone functions on $[a,b]$, then $\ell(gh) \leq \ell(g)\ell(h)$. Thus $\ell(f^n) \leq (\ell(f))^n$ for all $n \in \mathbb{N}$. Now fix a positive integer k. Given $n \in \mathbb{N}$, there are integers q and r such that $n = qk+r$ with $0 \leq r < k$. Then $\ell(f^n) \leq (\ell(f^k))^q(\ell(f))^r$, consequently

$$\sqrt[n]{\ell(f^n)} \leq (\ell(f^k))^{q/n}(\ell(f))^{r/n}.$$

Since k is fixed, $r/n \to 0$ and $q/n \to 1/k$ as $n \to \infty$. Therefore $\limsup \sqrt[n]{\ell(f^n)} \leq \sqrt[k]{\ell(f^k)}$ and we deduce that

$$\limsup \sqrt[n]{\ell(f^n)} \leq \inf \sqrt[k]{\ell(f^k)} \leq \liminf \sqrt[k]{\ell(f^k)}$$

and the result follows.