39th VTRMC, 2017, Solutions

1. Set $f(x) = 2x^6 - 6x^4 - 6x^3 + 12x^2 + 1 = 0$ and $g(x) = 2x^6 - 6x^4 - 4\sqrt{2}x^3 + 12x^2$. By raising to the sixth power, we see that a solution to the given equation also satisfies f. Furthermore to have a real solution, we need $x \leq \sqrt{2}$. Therefore if we can show that $f(x)$ has no solutions with $x \leq \sqrt{2}$, then it will follow that the original equation has no solutions. Now $g(x) = 2x^2(x-\sqrt{2})^2(x^2+2\sqrt{2}x+3)$. Thus g has zeros at 0 and $\sqrt{2}$ (of multiplicity 2), and is positive otherwise, because $x^2+2\sqrt{2}x+3 > 0$ for all $x \in \mathbb{R}$. Now $f(x) - g(x) = (4\sqrt{2}-6)x^3 + 1$ which is positive for $x \leq \sqrt{2}$, because the function is decreasing and $(4\sqrt{2}-6)\sqrt{2}^3 + 1 > 0$. To see this, we need to show that $17 - 12\sqrt{2} > 0$. However multiplying by $17 + 12\sqrt{2}$, we see that we need to show $17^2 - 144 \cdot 2 > 0$, which is true. It follows that the given equation has no real solutions.

2. Write $t = \tan(x/2)$. Then $\cos^2(x/2) = 1/(1+t^2)$, so
$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \frac{1-t^2}{1+t^2}$$
and since $\tan x = 2t/(1-t^2)$,
$$\sin x = \cos x \tan x = \frac{2t}{1+t^2}.$$
Write $I = \int_0^a \frac{dx}{1+\cos x+\sin x}$. Since $dt/dx = \frac{\sec^2(x/2)}{2} = (1+t^2)/2$, we see that
$$I = \int_0^{\tan(a/2)} \frac{2dt}{(1+t^2)+(1-t^2)+2t} = \int_0^{\tan(a/2)} \frac{dt}{1+t}.$$
Therefore $I = \ln(1+\tan(a/2))$. (An alternative answer is $\frac{1}{2}\ln\frac{1+\sin a}{1+\cos a} + \frac{1}{2}\ln 2$.) When $a = \pi/2$, we have $\tan(a/2) = 1$ and we deduce that $I = \ln 2$ as required.

3. We may assume that $AB = 1$. Since $\angle APB = 150$, the sine rule yields, $\sin 150/AB = \sin 20/AP = \sin 10/BP$ and $\sin 30/AP = \sin 40/CP$. Therefore $PC = 4\sin 20 \sin 40 = 2\cos 20 - 1$. Write $\angle PBC = \theta$. Since $\angle BPC =$

100, we see that $\angle PCB = 80 - \theta$, and then the sine rule for triangle BPC yields

$$\frac{2\cos 20 - 1}{\sin\theta} = \frac{2\sin 10}{\sin(80-\theta)} = \frac{2\sin 10}{\cos(\theta+10)}.$$

Therefore

$$2\cos 20\cos(\theta+10) = 2\sin 10\sin\theta + \cos(\theta+10) = \cos(\theta-10).$$

We deduce that $\cos(30+\theta)+\cos(10-\theta) = \cos(\theta-10)$ and hence $\cos(30+\theta) = 0$. We conclude that $\theta = 60$.

4. Denote the vertices of the triangle by A, B and C (counterclockwise). Let P be an interior point of the triangle and draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let EH be the segment parallel to AC, let FI be the segment parallel to BC, and let JG be the segment parallel AB. Here the points E, F lie on the edge AB; the points G, H lie on the edge BC, and the points I, J lie on the edge AC. Suppose that the area of the triangle EFP is a, the area of the triangle PGH is b, and the area of the triangle JPI is c. Note that the triangles EFP, PGH, JPI and ABC are similar. Therefore $EF/PG = \sqrt{a}/\sqrt{b}$ and $JP/PG = \sqrt{c}/\sqrt{b}$. Thus $(EF+JP)/PG = (\sqrt{a}+\sqrt{c})/\sqrt{b}$ and hence $1+(EF+JP)/PG = 1+(\sqrt{a}+\sqrt{c})/\sqrt{b}$, i.e.

$$\frac{PG+EF+JP}{PG} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{b}}.$$

Since $PG = FB$ and $JP = AE$, because $FBGP$ and $AEJP$ are parallelograms, $AB/PG = (\sqrt{a}+\sqrt{b}+\sqrt{c})/\sqrt{b}$. Because ABC is similar to PGH, we have $AB/PG = \sqrt{T}/\sqrt{b}$. Therefore $\sqrt{T} = \sqrt{a}+\sqrt{b}+\sqrt{c}$.

5. Let $(a,b) \in S$ and let $d = \gcd(a,b)$. Then $a = dm$ and $b = dn$ with $\gcd(m,n) = 1$. Since $g(a,b) \in \mathbb{N}$, we see that $ab = d^2mn$ is a perfect square and hence mn is a perfect square. Therefore m and n are both perfect squares, because $\gcd(m,n) = 1$. Thus we may write $a = ds^2$ and $b = dt^2$ with $\gcd(s,t) = 1$.

   By assumption, $h(a,b) = 2ds^2t^2/(s^2+t^2) \in \mathbb{N}$. Since $\gcd(s^2+t^2, s^2) = \gcd(s^2+t^2, t^2) = \gcd(s^2,t^2) = 1$, it follows that s^2+t^2 divides $2d$. Thus $a = k(s^2+t^2)s^2/2$ and $b = k(s^2+t^2)t^2/2$ for some $k \in \mathbb{N}$.

   Now $a \neq b$ because $s \neq \pm 1$. Also $f(a,b) = k(s^2+t^2)^2/4 \in \mathbb{N}$. We have two cases to consider.

- If s^2+t^2 is odd, then $4|k$ and hence $f(a,b) \geq 4(1^2+2^2)^2/4 = 25$.
- If s^2+t^2 is even, then s and t are odd because $\gcd(s,t)=1$ and hence $f(a,b) \geq (1^2+3^2)/4 = 25$.

We conclude that $f(a,b) \geq 25$. However $f(5,45) = f(10,40) = 25$, so the minimum of f over S is 25.

6. Set $g(x) = f(x) - x^2 + 4x - 2$. Then $g(1) = g(4) = g(8) = 0$. Therefore we may write $g(x) = (x-1)(x-4)(x-8)q(x)$ where $q(x) \in \mathbb{Z}[x]$. Since $f(n) = n^2 - 4n - 18$, we see that $g(n) = -20$ and hence $(n-1)(n-4)(n-8)q(n) = -20$. By inspection, $n = 3$ or 6. We note that both of these values of n can be obtained, by taking (for example) $q(x) = -2$ and 1 respectively, and then $f(x) = -2(x-1)(x-4)(x-8)+x^2-4x+2$ and $(x-1)(x-4)(x-8)+x^2-4x+2$ respectively.

7. First we look at small values of n: the given equation is a quadratic in m. If $n \in \{0,1,2,4\}$, there are no solutions. If $n = 3$, then $m = 6$ or 9. If $n = 5$, then $m = 9$ or 54. We now proceed by contradiction to show that there is no solution if $n \geq 6$. So suppose (m,n) is a solution with $n \geq 6$. Then m divides $2 \cdot 3^n$ and so either $m = 3^a$ for some $0 \leq a \leq n$, or $m = 2 \cdot 3^b$ for some $0 \leq b \leq n$. If $m = 3^a$, then

$$2^{n+1} - 1 = m + 2 \cdot 3^n/3^a = 3^a + 2 \cdot 3^{n-a}.$$

On the other hand if $m = 2 \cdot 3^b$, then

$$2^{n+1} - 1 = m + 2 \cdot 3^n/m = 2 \cdot 3^b + 3^{n-b}.$$

Therefore there must be nonnegative integers a, b such that

$$2^{n+1} - 1 = 3^a + 2 \cdot 3^b, \quad a + b = n.$$

Note that $3^a < 2^{n+1} < 3^{2(n+1)/3}$ and $2 \cdot 3^b < 2^{n+1} < 2 \cdot 3^{2(n+1)/3}$, because $3^{2/3} > 2$. Thus $a, b < 2(n+1)/3$. Since $a + b = n$, we deduce that

$$(n-2)/3 < a < 2(n+1)/3 \quad \text{and} \quad (n-2)/3 < b < 2(n+1)/3.$$

Now let $t = \min(a,b)$. Then $t > (n-2)/3$ and since $n \geq 6$, it follows that $t > 1$. Because 3^t divides 3^a and $2 \cdot 3^b$, we see that 3^t divides $2^{n+1} - 1$. Since

$t \geq 2$, we deduce that $2^{n+1} \equiv 1 \mod 9$. Now $2^{n+1} \equiv 1 \mod 9$ if and only if 6 divides $n+1$, so $n+1 = 6r$ for some $r \in \mathbb{N}$. Therefore

$$2^{n+1} - 1 = 4^{3r} - 1 = (4^{2r} + 4^r + 1)(4^r - 1) = (4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1).$$

Since 3^t divides $2^{n+1} - 1$, we see that 3^t divides $(4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1)$. Note that 9 does not divide $4^{2r} + 4^r + 1$, hence 3^{t-1} divides $(2^r - 1)(2^r + 1)$. Since $\gcd(2^r - 1,\ 2^r + 1) = 1$, either $3^{t-1} \mid 2^r - 1$ or $2^r + 1$. In any case, $3^{t-1} \leq 2^r + 1$. Then $3^{t-1} \leq 2^r + 1 \leq 3^r = 3^{(n+1)/6}$. Therefore $(n-2)/3 - 1 < t - 1 \leq (n+1)/6$. This yields $n < 11$, which is a contradiction, because $n \geq 6$ and we proved that $6 \mid n+1$.