37th VTRMC, 2015, Solutions

1. We have $f(n) := n^4 + 6n^3 + 11n^2 + 3n + 31 = (n^2 + 3n + 1)^2 - 3(n - 10)$. Therefore $f(10)$ is a perfect square, and we now show there is no other integer n such that $f(n)$ is a perfect square. We have $(n^2+3n+2)^2 - (n^2+3n+1)^2 = 2n^2+6n+3$ and $(n^2+3n+1)^2 - (n^2+3n)^2 = 2n^2+6n+1$. We have four cases to consider.

   (a) $n > 10$. Then we have $3(n-10) \geq 2n^2+6n+1$, which is not possible.

   (b) $2 < n < 10$. Then we have $3(10-n) \geq 2n^2+6n+3$, which is not possible.

   (c) $n < -6$. Then we have $3(10-n) \geq 2n^2+6n+3$, which is not possible.

   (d) $-6 \leq n \leq 2$. Then we can check individually that the 9 values of n do not make $f(n)$ a perfect square.

   We conclude that $f(n)$ is a perfect square only when $n = 10$.

2. The folded 3-dimensional region can be described as a regular tetrahedron with four regular tetrahedrons at each vertex cut off. The four smaller tetrahedrons have side length 2cm., while the big tetrahedron has sides of length 6cm. Recall that the volume of a regular tetrahedron of side of length 1 is $\sqrt{2}/12$ (or easy calculation). Therefore the volume required in cm^3 is

$$6^3\sqrt{2}/12 - 4 \cdot 2^3\sqrt{2}/12 = 46\sqrt{2}/3.$$

3. Let $n = 2015$. If one regards $a_1, \dots, a_n$ as variables, the determinant is skew symmetric (i.e. if we interchange a_i and a_j where $i \neq j$, we obtain $-\det A$). We deduce that $a_i - a_j$ divides $\det A$ for all $i \neq j$, hence

$$\det A \text{ is divisible by } \prod_{1 \leq i < j \leq n} (a_i - a_j).$$

   For $k \in \mathbb{N}$, we prove by induction on k that if a number is divisible $a_1 \cdots a_k$ and $\prod_{1 \leq i < j \leq k}(a_i - a_j)$, then it is divisible by $k!$; the case $k = 1$ is immediate. So assume the result for $i \leq k$. If one of the a_i is divisible by $k+1$, then the result is true for $k+1$ by induction. On the other hand if none of the a_i is divisible by $k+1$, then at least one of the numbers $a_i - a_j$ is divisible by $k+1$ and the induction step is complete. The result follows.

4. We first show the result is true if $0 < p \leq 1$ for $p \in \mathbb{Q}$ (positive number excludes 0, however the result is even true here by taking the sum of a zero number of terms). Write $p = a/b$ where $a, b \in \mathbb{N}$. The result is obviously true if $a = 1$. We now prove the result by induction on a; we may assume that $a < b$. Let $n \geq 2$ be the unique positive integer such that $1/n \leq p < 1/(n-1)$. Then we have $b \leq an$ and $0 \neq an - a < b$. Set $q = (an - b)/bn$. Since $an - b < a$, we may write q as a partial sum S of the $1/m$, and then we have $p = S + 1/n$. Also the integers m which appear in S must have $m > n$, because $p < n - 1$. This completes the induction step, and we have proven the result for $p \leq 1$.

   Let $p \in \mathbb{Q}$ and let $s_n = \sum_{k=1}^{n} 1/k$. Since the harmonic series is divergent, there exists a unique $m \in \mathbb{N}$ such that $s_m < p \leq s_{m+1}$. Then $p - s_m < 1$, so by the previous paragraph is a partial sum S of the $1/n$, and then we have $p = S + 1/m$. Also $S \leq 1/(m+1)$ so none of the $1/n$ appearing in S can be equal to $1/m$, and the proof is complete.

5. Let n be a positive integer. Then

$$\int_0^n \int_1^\pi \frac{1}{1+(xy)^2}\, dxdy = \int_1^\pi \int_0^n \frac{1}{1+(xy)^2}\, dydx.$$

   Therefore

$$\int_0^n \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \int_1^\pi \frac{\arctan(ny)}{y}\, dy.$$

   Set $u = \arctan(ny)$ and $dv = 1/y$ and use integration by parts to obtain

$$\int_1^\pi \frac{\arctan(ny)}{y}\, dy = \arctan(n\pi)\ln\pi - \int_1^\pi \frac{n\ln y}{1+n^2y^2}\, dy.$$

   On the other hand, $0 \leq \dfrac{n\ln y}{1+n^2} \leq \dfrac{n\ln\pi}{1+n^2}$ for all $y \in [1, \pi]$. Therefore

$$\lim_{n\to\infty} \int_1^\pi \frac{\arctan(ny)}{y}\, dy = \frac{\pi\ln\pi}{2}$$

   and we deduce that

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \lim_{n\to\infty} \int_0^n \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \frac{\pi\ln\pi}{2}.$$

6. If $(x,y) \in S := \sum_{i=1}^{n} \mathbb{Z}(a_i,b_i)$, then there exist $k_i \in \mathbb{Z}$ such that $(x,y) = \sum_{i=1}^{n} k_i(a_i,b_i)$. We choose the k_i such that $d := \sum_{i=1}^{n} |k_i|$ is minimal and then define $d(x,y) = d$. On the other hand if $(x,y) \notin S$, then define $d(x,y) = +\infty$ (thus $d(x,y) = \infty$ if and only if $(x,y) \notin S$). Now choose a positive integer m such that $m \geq n/\varepsilon$ and define

   $$f(x,y) = \begin{cases} 1 - d(x,y)/m & \text{if } d(x,y) \leq m; \\ 0 & \text{if } d(x,y) > m. \end{cases}$$

   If $(x,y) \notin S$, then $(x+a_i, y+b_i) \notin S$ for all i and therefore $d(x,y) = d(x+a_i, y+b_i) = 0$ and hence $f(x,y) = 0$ if $d(x,y) = +\infty$. On the other hand if $(x,y) \in S$, then $|d(x,y) - d(x+a_i, y+b_i)| \leq 1$ for all i. It then follows that $f(x,y) = 0$ if $d(x,y) > m$, hence $f(x,y) \neq 0$ for only finitely many (x,y), and furthermore $|f(x,y) - f(x+a_i, y+b_i)| = 0$ or $1/m$ for all i. Thus $f(x,y)$ satisfies the required condition, so the answer is "yes".

7. Note that the hypotheses show that there exists a positive integer a such that $a\langle u,v\rangle \in \mathbb{Z}$ for all $u,v \in S$. Therefore there exists a positive integer b such that $b\|u\|^2 = b\langle u,u\rangle$ is a positive integer for all $0 \neq u \in S$, so we may choose $0 \neq s \in S$ such that $\|s\|$ is minimal.

   First suppose that the x_i are all contained in $\mathbb{R}s$ (i.e. the points of S are collinear). Then the same is true of S and we claim that $S = \mathbb{Z}s$. If $u \in S$, then $u = cs$ for some $c \in \mathbb{R}$. Also $a\|s\| \leq \|u\| < (a+1)\|s\|$ for some nonnegative integer a, hence $\|as\| \leq \|cs\| < \|(a+1)s\|$. We deduce that $\|(a-c)s\| < \|s\|$ and since $(a-c)s \in S$, we conclude that $\|(a-c)s\| = 0$. Therefore $u = as$ and the claim is established. Now we place disks of radius $R := 3\|s\|/4$ and center $(2n+1/2)s$ for all $n \in \mathbb{Z}$ and the result is proven in this case.

   Now suppose that not all the x_i are not contained $\mathbb{R}s$. Then we may choose $t \in S \setminus \mathbb{R}s$ with $\|t\|$ minimal. We claim that $S = T := \{ms + nt \mid m,n \in \mathbb{Z}\}$. If this is not the case, we may choose $u \in S \setminus T$. Note that $\mathbb{R}s + \mathbb{R}t = \mathbb{R}^2$, so we may write $u = ps + qt$ for some $p,q \in \mathbb{R}$ and then there exist $a,b \in \mathbb{Z}$ such that $a \leq p < a+1$ and $b \leq q < b+1$, so u is inside the parallelogram with vertices (as,bt), $(as+s,bt)$, $(as,bt+t)$, $(as+s,bt+t)$. Since $\|s\| \leq \|t\|$ we see that u is distance at most $\|t\|$ from one of these vertices. Furthermore $\|u-v\| \geq \|t\|$ for all $u \neq v \in S$, so we must have $u \in S$.

   Now we can place disks with radius $R := \|s\|/2$ and centers at $((2m + 1/2)s, nt)$ for $m,n \in \mathbb{Z}$. Clearly every disk contains at least two points of S, namely $(2ms, nt)$ and $(2ms+1, nt)$ for the disk centered at $((2m+1/2)s, nt)$, and these disks accounts for all the points in S. We only need to show that any two distinct disks intersect in at most one point, and thus we need to show that two different centers are distance at least $\|s\|$ apart. So consider two different centers, say at $((2m+1/2)s, nt)$ and $((2m'+1/2)s, nt')$. Then the distance between theses two centers is the same at the distance between $(2ms, nt)$ and $(2m's, n't)$, which is at least $\|s\|$ by minimality of $\|s\|$. This completes the proof. (This actually proves the stronger statement that every point of S lies in exactly one disk, which is how the problem was meant to be stated; the argument can be significantly shortened for the actual problem.)