1. Let \( S \) denote the sum of the given series. By partial fractions,
\[
\frac{2n^2 - 2n - 4}{n^4 + 4n^2 + 16} = \frac{n - 2}{n^2 - 2n + 4} - \frac{n}{n^2 + 2n + 4}.
\]
If \( f(n) = \frac{n - 2}{n^2 - 2n + 4} \), then \( 2S = \sum_{n=2}^{\infty} f(n) - f(n+2) \). Since \( \lim_{n \to \infty} f(n) = 0 \), it follows by telescoping series that the series is convergent and \( 2S = f(2) - f(4) + f(3) - f(5) + f(4) - f(6) + \cdots \), so \( 2S = f(2) + f(3) \) and we deduce that \( S = 1/14 \).

2. Let \( I \) denote the given integral. First we make the substitution \( y = x^2 \), so \( dy = 2xdx \). Then
\[
2I = \int_0^4 \frac{16 - y}{\sqrt{16-y}(12+y)} \, dy = \int_0^4 \frac{\sqrt{16-y}}{\sqrt{16-y} + \sqrt{12+y}} \, dy.
\]
Now make the substitution \( z = 4 - y \), so \( dz = -dy \). Then
\[
2I = \int_0^4 \frac{\sqrt{12+z}}{\sqrt{12+z} + \sqrt{16-z}} \, dz.
\]
Adding the last two equations, we obtain \( 4I = \int_0^4 dz = 4 \) and hence \( I = 1 \).

3. Let \( m = \phi(2^{2014}) = 2^{2013} \) (here \( \phi(x) \) is Euler’s totient function, the number of positive integers \( < x \) which are prime to \( x \)). Then \( 19^m \equiv 1 \mod 2^{2014} \) by Euler’s theorem. Therefore \( n \) divides \( 2^{2013} \), so \( n = 2^k \) for some positive integer \( k \). Now
\[
19^{2^k} - 1 = (19 - 1)(19 + 1)(19^2 + 1)(19^4 + 1) \cdots (19^{2^{k-1}} + 1);
\]
we calculate the power of 2 in the above expression. This is \( 1 + 2 + 1 + \cdots + 1 = k + 2 \). Therefore \( k + 2 = 2014 \) and it follows that \( n = 2^{2012} \).

4. Put \( r^{a+2b} \) in the square in the \((a,b)\) position. Note that the sum of all the entries in a \( 4 \times 1 \) or \( 1 \times 4 \) rectangle is zero, because \( \sum_{k=0}^{3} r^{a+k+2b} = (1 + i + i^2 + i^3) = 0 \) and \( \sum_{k=0}^{3} r^{a+2(b+k)} = (1 + i^2 + i^4 + i^6) = 0 \). Therefore if we have a tiling with \( 4 \times 1 \) and \( 1 \times 4 \) rectangles, the sum of the entries in
all 361 squares is the value on the central square, namely \( i^{10+20} = -1 \). On the other hand, this sum is also

\[
(i + i^2 + \cdots + i^{19})(i^2 + i^4 + \cdots + i^{38}) = i^{\frac{i^{19} - 1}{i - 1} \cdot (-1 + 1 - \cdots - 1)} = i^{\frac{-i - 1}{i - 1} \cdot -1} = 1.
\]

This is a contradiction and therefore we have no such tiling.

5. Suppose by way of contradiction we can write \( n(n+1)(n+2) = m^r \), where \( n \in \mathbb{N} \) and \( r \geq 2 \). If a prime \( p \) divides \( n(n+2) \) and \( n+1 \), then it would have to divide \( n+1 \), and \( n \) or \( n+2 \), which is not possible. Therefore we may write \( n(n+2) = x' \) and \( n+1 = y' \) for some \( x, y \in \mathbb{N} \). But then \( n(n+2) + 1 = (n+1)^2 = z' \) where \( z = y^2 \). Since \( (n+1)^2 > n(n+2) \), we see that \( z > x \) and hence \( z \geq x+1 \), because \( x, z \in \mathbb{N} \). We deduce that \( z' \geq (x+1)r > x'+1 \), a contradiction and the result follows.

6. (a) Since \( A \) and \( B \) are finite subsets of \( T \), we may choose \( a \in A \) and \( b \in B \) so that \( f(ab) \) is as large as possible. Suppose we can write \( g := ab = cd \) with \( c \in A \) and \( d \in B \). Let \( h = d^{-1}b \) and \( d \neq b \). Note that \( g, h \in T \). Then \( h \neq I \) and we see that either \( f(gh^{-1}) > f(g) \) or \( f(gh) > f(g) \). This contradicts the maximality of \( f(ab) \). Therefore \( d = b \) and because \( b \) is an invertible matrix, we deduce that \( a = c \) and the result is proven.

(b) Set \( M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \). Then \( M \in S \) and \( M^3 = I \). Suppose \( f(M) > f(I) \).

Then \( (X = M \) and \( Y = M) \) we obtain either \( f(M^2) > f(M) \) or \( f(I) > f(M) \), hence \( f(M^2) > f(M) \). Now do the same with \( X = M^2 \) and \( Y = M \): we obtain \( f(M^3) > f(M^2) \). Since \( M^3 = I \), we now have \( f(I) > f(M^2) > f(M) > f(I) \), a contradiction. The argument is similar if we start out with \( f(M) < f(I) \). This shows that there is no such \( f \).

7. (a) Let \( A = (x_A, y_A) \) and \( B = (x_B, y_B) \). Then \( d(A, B) = \begin{pmatrix} x_B - x_A + y_A - y_B \\ x_B - x_A \end{pmatrix} \).

(b) By definition \( \det M = d(A_1, B_1)d(A_2, B_2) - d(A_1, B_2)d(A_2, B_1) \). Note that the first term counts all pairs of paths \( (\pi_1, \pi_2) \) where \( \pi_i : A_i \to B_i \), and the second term is the negative of the number of pairs \( (\pi_1, \pi_2) \) where \( \pi_1 : A_1 \to B_2 \) and \( \pi_2 : A_2 \to B_1 \). The configuration of the points implies that every pair of paths \( (\pi_1, \pi_2) \) where \( \pi_1 : A_1 \to B_2 \) and \( \pi_2 : A_2 \to B_1 \)
must intersect. Let $\mathcal{I} := \{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 \neq \emptyset\}$ (this is the set of all intersecting paths, regardless of their endpoints). Define $\Phi : \mathcal{I} \to \mathcal{I}$ as follows. If $(\pi_1, \pi_2) \in \mathcal{I}$ then $\Phi((\pi_1, \pi_2)) = (\pi'_1, \pi'_2)$ and the new pair of paths is obtained from the old one by switching the tails of $\pi_1, \pi_2$ after their last intersection point. In particular, the pairs $(\pi_1, \pi_2)$ and $(\pi'_1, \pi'_2)$ must appear in different terms of $\det M$. But it is clear that $\Phi \circ \Phi = id_{\mathcal{I}}$, therefore $\Phi$ is an involution. This implies that all intersecting pairs of paths must cancel each other, and that the only pairs which contribute to the determinant are those from the set $\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$. Since all the latter pairs can appear only with positive sign (in the first term of $\det M$), this finishes the solution. (In fact, we proved that $\det M = \#\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$.)