36th VTRMC, 2014, Solutions

1. Let S denote the sum of the given series. By partial fractions,

$$2\frac{n^2-2n-4}{n^4+4n^2+16}=\frac{n-2}{n^2-2n+4}-\frac{n}{n^2+2n+4}.$$

If $f(n)=\frac{n-2}{n^2-2n+4}$, then $2S=\sum_{n=2}^{n=\infty} f(n)-f(n+2)$. Since $\lim_{n\to\infty} f(n)=0$, it follows by telescoping series that the series is convergent and $2S=f(2)-f(4)+f(3)-f(5)+f(4)-f(6)+\cdots$, so $2S=f(2)+f(3)$ and we deduce that $S=1/14$.

2. Let I denote the given integral. First we make the substitution $y=x^2$, so $dy=2xdx$. Then

$$2I=\int_0^4 \frac{16-y}{16-y+\sqrt{(16-y)(12+y)}}\,dy=\int_0^4 \frac{\sqrt{16-y}}{\sqrt{16-y}+\sqrt{12+y}}\,dy.$$

Now make the substitution $z=4-y$, so $dz=-dy$. Then

$$2I=\int_0^4 \frac{\sqrt{12+z}}{\sqrt{12+z}+\sqrt{16-z}}\,dz.$$

Adding the last two equations, we obtain $4I=\int_0^4 dz=4$ and hence $I=1$.

3. Let $m=\phi(2^{2014})=2^{2013}$ (here $\phi(x)$ is Euler's totient function, the number of positive integers $<x$ which are prime to x). Then $19^m\equiv 1 \mod 2^{2014}$ by Euler's theorem. Therefore n divides 2^{2013}, so $n=2^k$ for some positive integer k. Now

$$19^{2^k}-1=(19-1)(19+1)(19^2+1)(19^4+1)\ldots(19^{2^{k-1}}+1);$$

we calculate the power of 2 in the above expression. This is $1+2+1+1+\cdots+1=k+2$. Therefore $k+2=2014$ and it follows that $n=2^{2012}$.

4. Put i^{a+2b} in the square in the (a,b) position. Note that the sum of all the entries in a 4×1 or 1×4 rectangle is zero, because $\sum_{k=0}^{3} i^{a+k+2b}=(1+i+i^2+i^3)i^{a+2b}=0$ and $\sum_{k=0}^{3} i^{a+2(b+k)}=(1+i^2+i^4+i^6)i^{a+2b}=0$. Therefore if we have a tiling with 4×1 and 1×4 rectangles, the sum of the entries in

all 361 squares is the value on the central square, namely $i^{10+20} = -1$. On the other hand this sum is also

$$\begin{aligned}(i+i^2+\cdots+i^{19})(i^2+i^4+\cdots+i^{38}) &= i\frac{i^{19}-1}{i-1}\cdot(-1+1-\cdots-1)\\ &= i\frac{-i-1}{i-1}\cdot -1 = 1.\end{aligned}$$

This is a contradiction and therefore we have no such tiling.

5. Suppose by way of contradiction we can write $n(n+1)(n+2) = m^r$, where $n \in \mathbb{N}$ and $r \geq 2$. If a prime p divides $n(n+2)$ and $n+1$, then it would have to divide $n+1$, and n or $n+2$, which is not possible. Therefore we may write $n(n+2) = x^r$ and $n+1 = y^r$ for some $x, y \in \mathbb{N}$. But then $n(n+2)+1 = (n+1)^2 = z^r$ where $z = y^2$. Since $(n+1)^2 > n(n+2)$, we see that $z > x$ and hence $z \geq x+1$, because $x, z \in \mathbb{N}$. We deduce that $z^r \geq (x+1)^r > x^r + 1$, a contradiction and the result follows.

6. (a) Since A and B are finite subsets of T, we may choose $a \in A$ and $b \in B$ so that $f(ab)$ is as large as possible. Suppose we can write $g := ab = cd$ with $c \in A$ and $d \in B$. Let $h = d^{-1}b$ and $d \neq b$. Note that $g, h \in T$. Then $h \neq I$ and we see that either $f(gh^{-1}) > f(g)$ or $f(gh) > f(g)$. This contradicts the maximality of $f(ab)$. Therefore $d = b$ and because b is an invertible matrix, we deduce that $a = c$ and the result is proven.

   (b) Set $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then $M \in S$ and $M^3 = I$. Suppose $f(M) > f(I)$. Then ($X = M$ and $Y = M$) we obtain either $f(M^2) > f(M)$ or $f(I) > f(M)$, hence $f(M^2) > f(M)$. Now do the same with $X = M^2$ and $Y = M$: we obtain $f(M^3) > f(M^2)$. Since $M^3 = I$, we now have $f(I) > f(M^2) > f(M) > f(I)$, a contradiction. The argument is similar if we start out with $f(M) < f(I)$. This shows that there is no such f.

7. (a) Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Then $d(A,B) = \begin{pmatrix} x_B - x_A + y_A - y_B \\ x_B - x_A \end{pmatrix}$.

   (b) By definition $\det M = d(A_1,B_1)d(A_2,B_2) - d(A_1,B_2)d(A_2,B_1)$. Note that the first term counts all pairs of paths (π_1, π_2) where $\pi_i : A_i \to B_i$, and the second term is the negative of the number of pairs (π_1, π_2) where $\pi_1 : A_1 \to B_2$ and $\pi_2 : A_2 \to B_1$. The configuration of the points implies that every pair of paths (π_1, π_2) where $\pi_1 : A_1 \to B_2$ and $\pi_2 : A_2 \to B_1$

must intersect. Let $\mathscr{I} := \{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 \neq \emptyset\}$ (this is the set of all intersecting paths, regardless of their endpoints). Define $\Phi\colon \mathscr{I} \to \mathscr{I}$ as follows. If $(\pi_1, \pi_2) \in \mathscr{I}$ then $\Phi((\pi_1, \pi_2)) = (\pi_1', \pi_2')$ and the new pair of paths is obtained from the old one by switching the tails of π_1, π_2 after their *last* intersection point. In particular, the pairs (π_1, π_2) and (π_1', π_2') must appear in different terms of $\det M$. But it is clear that $\Phi \circ \Phi = id_{\mathscr{I}}$, therefore Φ is an involution. This implies that all intersecting pairs of paths must cancel each other, and that the only pairs which contribute to the determinant are those from the set $\{(\pi_1, \pi_2)\colon\ \pi_1 \cap \pi_2 = \emptyset\}$. Since all the latter pairs can appear only with positive sign (in the first term of $\det M$), this finishes the solution. (In fact, we proved that $\det M = \#\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$.)