35th VTRMC, 2013, Solutions

1. Make the substitution $t = 2y$, so $dt = 2dy$. Then $I = \int_0^{x/2} 6\sqrt{2}\frac{\sqrt{(1+1\cos 2y)/2}}{17-8\cos 2y}dy = \int_0^{x/2} 3\sqrt{2}\frac{2\sqrt{2}\cos y}{9+16\sin^2 y}dy$. Now make the substitution $z = \sin y$. Then $dz = dy\cos y$ and $I = 12\int_0^{\sin x/2}\frac{dz}{3^2+4^2z^2} = \tan^{-1}\frac{4}{3}\sin x/2$. If $\tan I = 2/\sqrt{3}$, then $2\sqrt{3} = \frac{4}{3}\sin x/2$ and we deduce that $x = 2\pi/3$.

2. Without loss of generality we may assume that $BC = 1$, and then we set $x := BD$, so $AD = 2x$. Write $\theta = \angle CAD$, $y = AC$ and $z = DC$. The area of ADC is both x and $(yz\sin\theta)/2$. Also $y^2 = 1+9x^2$ and $z^2 = 1+x^2$. Therefore $4x^2 = (1+9x^2)(1+x^2)\sin^2\theta$. We need to maximize θ, equivalently $\sin^2\theta$, which in turn is equivalent to minimizing $(1+9x^2)(1+x^2)/(4x^2)$. Therefore we need to find x such that $x^{-2}+9x^2$ is minimal. Differentiating, we find $-2x^{-3}+18x = 0$, so $x^2 = 1/3$. It follows that $\sin^2\theta = 1/4$ and we deduce that the maximum value of $\angle CAD = \theta$ is $30°$.

3. We need to show that a_n is bounded, equivalently $\ln a_n$ is bounded, i.e. $\ln 2\sum_{n=1}^{\infty}\ln(1+n^{-3/2})$ is bounded. But $\ln(1+n^{-3/2}) < n^{-3/2}$ and $\sum_{n=1}^{\infty} n^{-3/2}$ is convergent. It follows that (a_n) is convergent.

4. (a) $25 = 50/2 = \dfrac{5^2+5^2}{1^2+1^2}$.

   (b) Assume that 2013 is special. Then we have
$$x^2+y^2 = 2013(u^2+v^2) \tag{1}$$
   for some positive integers x,y,u,v. Also, we assume that x^2+y^2 is minimal with this property. The prime factorization of 2013 is $3\cdot 11\cdot 61$. From (1) it follows $3|x^2+y^2$. It is easy to check by looking to the residues mod 3 that $3|x$ and $3|y$, hence we have $x = 3x_1$ and $y = 3y_1$. Replacing in (1) we get
$$3(x_1^2+y_1^2) = 11\cdot 61(u^2+v^2), \tag{2}$$
   i.e. $3|u^2+v^2$. It follows $u = 3u_1$ and $v = 3v_1$, and replacing in (2) we get
$$x_1^2+y_1^2 = 2013(u_1^2+v_1^2).$$
   Clearly, $x_1^2+y_1^2 < x^2+y^2$, contradicting the minimality of x^2+y^2.

(c) Observe that $2014 = 2 \cdot 19 \cdot 53$ and 19 is a prime of the form $4k+3$. If 2014 is special, then we have,

$$x^2 + y^2 = 2014(u^2 + v^2),$$

for some positive integers x, y, u, v. As in part (b), we may assume that $x^2 + y^2$ is minimal with this property. Now, we will use the fact that if a prime p of the form $4k+3$ divides $x^2 + y^2$, then it divides both x and y. Indeed, if p does not divide x, then it does not divide y too. We have $x^2 \equiv -y^2 \mod p$ implies $(x^2)^{\frac{p-1}{2}} \equiv (-y^2)^{\frac{p-1}{2}} \mod p$. Because $\frac{p-1}{2} = 2k+1$, the last relation is equivalent to $(x^2)^{\frac{p-1}{2}} \equiv -(y^2)^{\frac{p-1}{2}} \mod p$, hence $x^{p-1} \equiv -y^{p-1} \mod p$. According to the Fermat's little theorem, we obtain $1 \equiv -1 \mod p$, that is p divides 2, which is not possible.

Now continue exactly as in part (b) using the prime 19, and contradict the minimality of $x^2 + y^2$.

5. Write $x = \tan A$, $y = \tan B$, $z = \tan C$, where $0 < A, B, C < \pi/2$. Using the formula $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ twice, we see that

$$\tan(A+B+C) = \frac{x+y+z-xyz}{1-yz-zx-xy} = 0$$

and therefore $A+B+C = \pi$. Now $\sin A = \frac{x}{1+x^2}$, so we need to prove that $\sin A + \sin B + \sin C \leq 3\sqrt{3}/2$. However $\sin t$ is a concave function, so we may apply Jensen's inequality (or consider the tangent at $t = (A+B+C)/3$) to deduce that

$$\frac{\sin A + \sin B + \sin C}{3} \leq \frac{\sin(A+B+C)}{3} = \sin(\pi/3) = \sqrt{3}/2,$$

and the result follows.

6. Let $C = X^{-1} + (Y^{-1} - X)^{-1}$. Observe that $(Y^{-1} - X) = (X - XYX)X^{-1}Y^{-1}$, consequently $(Y^{-1} - X)^{-1} = YX(X - XYX)^{-1}$. Therefore $C(X - XYX)^{-1} = I - YX + YX = I$ and we deduce that $XY - BY = (X - X + XYXD)Y = XYXY$. Therefore we can take $M = XY = \begin{pmatrix} 190 & 81 & 65 \\ -49 & 64 & -191 \\ -56 & 74 & 86 \end{pmatrix}$.

7. For $|q| < 1$, we have $\sum_{k=1}^{\infty} q^k = q/(q-1)$. Therefore for $|q| > 1$,

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} &= -\sum_{n=1}^{\infty} \frac{(-1)^n q^{-n}}{1 - q^{-n}} \\
&= -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (q^{-n})^k (-1)^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{-n}}{1 - q^{-n}}, \\
\sum_{n=1}^{\infty} \frac{1}{q^n + 1} &= \sum_{n=1}^{\infty} \frac{q^{-n}}{1 + q^{-n}} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} (q^{-n})^k \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{-k}}{1 - q^{-k}}.
\end{aligned}$$

It follows that $-\sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{1}{q^n + 1}$. Now

$$\begin{aligned}
\frac{d}{dx} \frac{1}{x^n - 1} &= \frac{-n}{x(x^{n/2} - x^{-n/2})^2} \\
\frac{d}{dx} \frac{1}{x^n + 1} &= \frac{-n}{x(x^{n/2} + x^{-n/2})}.
\end{aligned}$$

We deduce that

$$-\sum_{n=1}^{\infty} \frac{(-1)^n n}{q(q^{n/2} - q^{-n/2})^2} = \sum_{n=1}^{\infty} \frac{n}{q(q^{n/2} + q^{-n/2})^2}$$

Now set $q = 4$. We conclude that $\sum_{n=1}^{\infty} \frac{n}{(2^n + 2^{-n})^2} + \frac{(-1)^n n}{(2^n - 2^{-n})^2} = 0.$