1. Make the substitution $t = 2y$, so $dt = 2dy$. Then $I = \int_0^{\pi/2} 6\sqrt{2} \frac{\sqrt{(1 + \cos 2y)^2}}{17 - 8\cos 2y} \, dy = \int_0^{\pi/2} 3\sqrt{2} \frac{2\sqrt{2}\cos y}{9 + 16\sin^2 y} \, dy$. Now make the substitution $z = \sin y$. Then $dz = dy \cos y$ and $I = 12 \int_0^{\pi/2} \frac{3z}{3^2 + 4z^2} = \tan^{-1} \frac{\sqrt{3}}{3} \sin x/2$. If $\tan I = 2/\sqrt{3}$, then $2\sqrt{3} = 4\sin x/2$ and we deduce that $x = 2\pi/3$.

2. Without loss of generality we may assume that $BC = 1$, and then we set $x := BD$, so $AD = 2x$. Write $\theta = \angle CAD$, $y = AC$ and $z = DC$. The area of $ADC$ is both $x$ and $(yz\sin \theta)/2$. Also $y^2 = 1 + 9x^2$ and $z^2 = 1 + x^2$. Therefore $4x^2 = (1 + 9x^2)(1 + x^2)\sin^2 \theta$. We need to maximize $\theta$, equivalently $\sin^2 \theta$, which in turn is equivalent to minimizing $(1 + 9x^2)(1 + x^2)/(4x^2)$. Therefore we need to find $x$ such that $x^{-2} + 9x^2$ is minimal. Differentiating, we find $-2x^{-3} + 18x = 0$, so $x^2 = 1/3$. It follows that $\sin^2 \theta = 1/4$ and we deduce that the maximum value of $\angle CAD = \theta$ is $30^\circ$.

3. We need to show that $a_n$ is bounded, equivalently $\ln a_n$ is bounded, i.e. $\ln 2 \sum_{n=1}^{\infty} \ln (1 + n^{-3/2})$ is bounded. But $\ln (1 + n^{-3/2}) < n^{-3/2}$ and $\sum_{n=1}^{\infty} n^{-3/2}$ is convergent. It follows that $(a_n)$ is convergent.

4. (a) $25 = 50/2 = \frac{5^2 + 5^2}{1^2 + 1^2}$.

(b) Assume that 2013 is special. Then we have

$$x^2 + y^2 = 2013(u^2 + v^2) \quad (1)$$

for some positive integers $x, y, u, v$. Also, we assume that $x^2 + y^2$ is minimal with this property. The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. From (1) it follows $3|x^2 + y^2$. It is easy to check by looking to the residues mod 3 that $3|x$ and $3|y$, hence we have $x = 3x_1$ and $y = 3y_1$. Replacing in (1) we get

$$3(x_1^2 + y_1^2) = 11 \cdot 61(u^2 + v^2), \quad (2)$$

i.e. $3|u^2 + v^2$. It follows $u = 3u_1$ and $v = 3v_1$, and replacing in (2) we get

$$x_1^2 + y_1^2 = 2013(u_1^2 + v_1^2).$$

Clearly, $x_1^2 + y_1^2 < x^2 + y^2$, contradicting the minimality of $x^2 + y^2$. 

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(c) Observe that \(2014 = 2 \cdot 19 \cdot 53\) and 19 is a prime of the form \(4k + 3\). If 2014 is special, then we have,
\[
x^2 + y^2 = 2014(u^2 + v^2),
\]
for some positive integers \(x, y, u, v\). As in part (b), we may assume that \(x^2 + y^2\) is minimal with this property. Now, we will use the fact that if a prime \(p\) of the form \(4k + 3\) divides \(x^2 + y^2\), then it divides both \(x\) and \(y\). Indeed, if \(p\) does not divide \(x\), then it does not divide \(y\) too. We have \(x^2 \equiv -y^2 \pmod{p}\) implies \((x^2)^{p-1} \equiv (-y^2)^{p-1} \pmod{p}\). Because \(p-1 = 2k + 1\), the last relation is equivalent to \((x^2)^{p-1} \equiv -(y^2)^{p-1} \pmod{p}\), hence \(x^{p-1} \equiv -y^{p-1} \pmod{p}\). According to the Fermat's little theorem, we obtain \(1 \equiv -1 \pmod{p}\), that is \(p\) divides 2, which is not possible.

Now continue exactly as in part (b) using the prime 19, and contradict the minimality of \(x^2 + y^2\).

5. Write \(x = \tan A\), \(y = \tan B\), \(z = \tan C\), where \(0 < A, B, C < \pi/2\). Using the formula \(\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\), twice, we see that
\[
\tan(A + B + C) = \frac{x + y + z - xyz}{1 - yz - zx - xy} = 0
\]
and therefore \(A + B + C = \pi\). Now \(\sin A = \frac{x}{1 + x^2}\), so we need to prove that \(\sin A + \sin B + \sin C \leq 3\sqrt{3}/2\). However \(\sin t\) is a concave function, so we may apply Jensen's inequality (or consider the tangent at \(t = (A + B + C)/3\)) to deduce that
\[
\frac{\sin A + \sin B + \sin C}{3} \leq \frac{\sin(A + B + C)}{3} = \sin(\pi/3) = \sqrt{3}/2,
\]
and the result follows.

6. Let \(C = X^{-1} + (Y^{-1} - X)^{-1}\). Observe that \((Y^{-1} - X) = (X - XYX)X^{-1}Y^{-1}\), consequently \((Y^{-1} - X)^{-1} = YX(X - XYX)^{-1}\). Therefore \(C(X - XYX)^{-1} = I - YX + YX = I\) and we deduce that \(XY - BY = (X - X + XYXD)Y = XYXY\). Therefore we can take \(M = XY = \begin{pmatrix} 190 & 81 & 65 \\ -49 & 64 & -191 \\ -56 & 74 & 86 \end{pmatrix}\).
7. For $|q| < 1$, we have $\sum_{k=1}^{\infty} q^k = q/(q - 1)$. Therefore for $|q| > 1$,

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = -\sum_{n=1}^{\infty} \frac{(-1)^n q^{-n}}{1 - q^{-n}} = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (q^{-n})^k (-1)^{n+1} = \sum_{n=1}^{\infty} (\frac{-1}{1 - q^{-n}})^{n+1}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{q^n + 1} = \sum_{n=1}^{\infty} \frac{q^{-n}}{1 + q^{-n}} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} (q^{-n})^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{-k}}{1 - q^{-k}}.
\]

It follows that $-\sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{1}{q^n + 1}$. Now

\[
\frac{d}{dx} \frac{1}{x^n - 1} = \frac{-n}{x(x^{n/2} - x^{-n/2})^2}
\]

\[
\frac{d}{dx} \frac{1}{x^n + 1} = \frac{-n}{x(x^{n/2} + x^{-n/2})^2}.
\]

We deduce that

\[
-\sum_{n=1}^{\infty} \frac{(-1)^n}{q(q^{n/2} - q^{-n/2})^2} = \sum_{n=1}^{\infty} \frac{n}{q(q^{n/2} + q^{-n/2})^2}
\]

Now set $q = 4$. We conclude that $\sum_{n=1}^{\infty} \frac{n}{(2^n + 2^{-n})^2} + \frac{(-1)^n}{(2^n - 2^{-n})^2} = 0$. 