34th VTRMC, 2012, Solutions

1. Let I denote the value of the integral. We make the substitution $y = \pi/2 - x$. Then $dx = -dy$, and as x goes from 0 to $\pi/2$, y goes from $\pi/2$ to 0. Also $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$. Thus

$$I = \int_0^{\pi/2} \frac{\sin^4 x + \sin x\ \cos^3 x + \sin^2 x\ \cos^2 x + \sin^3 x\ \cos x}{\sin^4 x + \cos^4 x + 2\sin x\ \cos^3 x + 2\sin^2 x\ \cos^2 x + 2\sin^3 x\ \cos x}\, dx.$$

Adding the above to the given integral, we obtain $2I = \int_0^{\pi/2} dx$. Therefore $I = \pi/4$.

2. We necessarily have $x \geq -2$. Also the left hand side becomes negative for $x \geq 2$. Therefore we may assume that $x = 2\cos t$ for $0 \leq t \leq \pi$. After making this substitution, the equation becomes $\cos 3t + \cos(t/2) = 0$. Using a standard trig formula ($2\cos A \cos B = \cos(A+B) + \cos(A-B)$), this becomes $\cos(7t/4)\cos(5t/4) = 0$. This results in the solutions $t = 2\pi/5$, $2\pi/7$, $6\pi/7$. Therefore $x = 2\cos(2\pi/5)$, $2\cos(2\pi/7)$, $2\cos(6\pi/7)$.

3. We make a, b, c, d, e be the roots of the quintic equation $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$. Using the first and last equations, we get $p = t = 1$. Let $Z = \{a, b, c, d, e\}$. Then

$$2q = \sum_{u,v \in Z,\ u \neq v} uv = (a + \cdots + e)^2 - (a^2 + \cdots + e^2) = -14,$$

so $q = -7$. Next $s = abcde(1/a + \cdots + 1/e) = -1/-1 = 1$. Finally

$$r = abcde(\sum_{u,v \in Z,\ u \neq v} uv) = abcde\big((1/a + \cdots + 1/e)^2 - (1/a^2 + \cdots + 1/e^2)\big)$$
$$= -14,$$

so $r = -7$.

Similarly $s = 1$ and $r = -7$. Therefore a, b, c, d, e are the roots of $x^5 + x^4 - 7x^3 - 7x^2 + x + 1 = 0$. By inspection, -1 is a root and the equation factors as

$$(x+1)(x^4 - 7x^2 + 1) = (x+1)(x^2 - 3x + 1)(x^2 + 3x + 1).$$

Using the quadratic formula, it follows that a, b, c, d, e are (in whatever order you like)

$$-1, \frac{\pm 3 \pm \sqrt{5}}{2}.$$

4. We repeatedly use the fact that if n is a positive integer and $a \in \mathbb{Z}$ is prime to n, then $a^{\phi(n)} \equiv 1 \bmod n$ where ϕ is Euler's totient function.

   We first show that $f(n) \equiv 3 \bmod 4$ for all $n \geq 1$. We certainly have $f(1) \equiv 3 \bmod 4$. Since $f(n)$ is always odd, we see that $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 4$ and we deduce that $f(n) \equiv 3 \bmod 4$ for all $n \geq 1$.

   Now we show that $f(n) \equiv f(3) \bmod 25$ for all $n \geq 3$. First observe that $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 5$ for $n \geq 2$, provided $f(n) = f(n-1) \bmod 4$, which is true by the previous paragraph. It follows that $f(n+1) \equiv f(n) \bmod 20$ for all $n \geq 2$. Therefore $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 25$, provided $n \geq 3$, and our assertion is proven. Since the last two digits of $f(3)$ are 87, the last two digits of $f(2012)$ are also 87.

5. Let $f(n) = 1/(\ln n) - (1/\ln n)^{(n+1)/n}$. Then $f(n)\ln n = 1 - (\ln n)^{-1/n}$. Assume that $n > 27$. Since $\ln n > e$, we see that $f(n)\ln n > 1 - e^{-1/n}$. Therefore $nf(n)\ln n > n(1 - e^{-1/n})$. By L'hôpital's rule, $\lim_{n\to\infty} n(1 - e^{-1/n}) = 1$. Therefore $nf(n)\ln n > 1/2$ for sufficiently large n, so $f(n) > 1/(2n\ln n)$. Since $\sum_{n=2}^{\infty} 1/(n\ln n)$ is divergent, it follows that $\sum_{n=2}^{\infty} f(n)$ is also divergent.

6. We shall prove by induction that $a_n = p$ if p is a prime and $n = p^m$ for some positive integer m, and 1 otherwise. This is clear in the case $n = p^m$, because then there are exactly $m-1$ nontrivial divisors of p^m, and each contributes p to the denominator of the displayed fraction. The case $n = pq$, where p, q are distinct primes, is also clear, because then p and q are the only nontrivial divisors of n, and they contribute p and q respectively to the denominator.

   Now assume that n is neither a prime power, nor a product of two distinct primes, and assume the result is true for all smaller values of n. Then we may write $n = pm$, where p is a prime and m is not a prime power. Write $m = p^a k$, where a is a nonnegative integer and k is prime to p. If $d \mid n$, then either $d \mid m$ or $d = p^{a+1} r$, where $r \mid k$. Note that in the latter case, d is a prime power only when $r = 1$. Therefore

   $$a_n = a_m \frac{p}{a_{p^{a+1}}} = 1\frac{p}{p} = 1.$$

   by induction, which proves the claim. It follows that $a_{999000} = 1$.

7. Let 0 and I denote the zero and identity 2×2 matrices respectively. Let A denote one of the three matrices. The result is clear if $A = 0$ or I, because

every matrix commutes with 0 and I. Next note that $A^i = A^j$, where $0 < i < j < 5$, and we see that the minimum polynomial of A divides $x^j - x^i$.

Suppose 0 is not an eigenvalue of A. Then A is invertible and it follows that $A^{j-i} = I$, in particular I is one of the matrices. Since I commutes with all matrices, the result follows in this case.

Thus we may assume that 0 is an eigenvalue of A. Next suppose both the eigenvalues of A are 0 (i.e. A has a repeated eigenvalue 0). Then the Jordan canonical form of A is either 0 or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In both case, $A^2 = 0$, and since 0 commutes with all matrices, the result follows in this case.

Therefore we may assume that A has one eigenvalue 0 and another eigenvalue $\lambda \neq 0$. Since the minimum polynomial of A divides $x^j - x^i$ where $0 < i < j < 5$, we see that the possibilities for λ are 1, -1, or ω where ω is a primitive cube root of 1. Since the eigenvalues of A are distinct, it is diagonalizable and in particular, its Jordan canonical form will be $\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$. If $\lambda = \omega$, then $\{A, A^2, A^3\}$ are three distinct commuting matrices, and the result is proven in this case. Thus we may assume that the Jordan canonical form for A is $\begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Now not all the A_i can have Jordan canonical form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, because then $\operatorname{tr}(A_1 + A_2 + A_3) = 3$, so at least one of the matrices, say A_1, has trace -1. It should be pointed out that we may assume that the A_i are distinct, if not, then the three matrices come from $\{A_1, A_1^2\}$, and since A_1 commutes with A_1^2, the result follows in this case.

Suppose $\operatorname{tr}(A_2) = -1$ and $\operatorname{tr}(A_3) = 1$. Then $A_1^2 = A_2^2 = A_3$ and A_3 commutes with A_1 and A_2, and the result is proven in this case.

Finally suppose $\operatorname{tr}(A_2) = \operatorname{tr}(A_3) = 1$. Then without loss of generality, we may assume that $A_1^2 = A_2$, and so $A_1 = -A_2$. Thus $-A_3 \neq A_1$ or A_2. Since $A_2A_3 = -A_1A_3$, we see that $A_2A_3 = A_1$ or A_2. Similarly $A_3A_2 = A_1$ or A_2. Since $\operatorname{tr}(A_2A_3) = \operatorname{tr}(A_3A_2)$, we deduce that $A_2A_3 = A_3A_2$. Also $A_1A_2 = A_2A_1$, and the result follows.