# 30th VTRMC, 2008, Solutions

1. Write $f(x) = xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$. First we look for local maxima, so we need to solve $\partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0$. Now $\partial f/\partial x = y^3 + 3x^2z - z^3 - 3x^2y$. If $y = z$, then $f(x,y,z) = 0$ and this is not a maximum. Thus we may divide by $y - z$ and then $\partial f/\partial x = 0$ yields $y^2 + yz + z^2 = 3x^2$. Similarly $x^2 + xz + z^2 = 3y^2$ and $x^2 + xy + y^2 = 3z^2$. Adding these three equations, we obtain $(x-y)^2 + (y-z)^2 + (z-x)^2 = 0$, which yields $x = y = z$. This does not give a maximum, because $f = 0$ in this case, and we conclude that the maximum of $f$ must occur on the boundary of the region, so at least one of $x, y, z$ is 0 or 1.

   Let's look at $f$ on the side $x = 0$. Here $f = yz^3 - y^3z$ and $0 \le y \le 1$, $0 \le z \le 1$. To find local maxima, we solve $\partial f/\partial y = \partial f/\partial z = 0$. This yields $y = z = 0$ and $f = 0$, which is not a maximum, so the maximum occurs on the edges of the region considered. If $y$ or $z = 0$, we get $f = 0$ which is not a maximum. If $y = 1$, then $f = z^3 - z \le 0$, which won't give a maximum. Finally if $z = 1$, then $f = y - y^3$. Since $df/dy = 1 - 3y^2$, we see that $f$ has a maximum at $y = 1/\sqrt{3}$. This gives that the maximum value of $f$ on $x = 0$ is $1/\sqrt{3} - 1/\sqrt{3}^3 = 2\sqrt{3}/9$.

   Similarly if $y$ or $z = 0$, the maximum value of $f$ is $2\sqrt{3}/9$. Now let's look at $f$ on the side $x = 1$. Here $f = y^3 + yz^3 + z - y - y^3z - z^3$. Again we first look for local maxima: $\partial f/\partial y = 3y^2 + z^3 - 1 - 3y^2z$. Then $\partial f/\partial y = 0$ yields either $z = 1$ or $3y^2 = z^2 + z + 1$. If $z = 1$, then $f = 0$ which is not a maximum, so $3y^2 = z^2 + z + 1$. Similarly $3z^2 = y^2 + y + 1$. Adding these two equations, we find that $y^2 - y/2 + z^2 - z/2 = 1$. Thus $(y - 1/2)^2 + (z - 1/2)^2 = 3/2$. This has no solution in the region considered $0 \le y \le 1, 0 \le z \le 1$. Thus $f$ must have a maximum on one of the edges. If $y$ or $z$ is 0, then we are back in the previous case. On the other hand if $y$ or $z$ is 1, then $f = 0$, which is not a maximum.

   We conclude that the maximum value of $f$ on $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ is $2\sqrt{3}/9$.

2. For each positive integer $n$, let $f(n)$ denote the number of sequences of 1's and 3's that sum to $n$. Then $f(n+3) = f(n+2) + f(n)$, and we have $f(1) = 1$, $f(2) = 1$, and $f(3) = 2$. Thus $f(4) = f(3) + f(1) = 3$, $f(5) = f(4) + f(2) = 4$, $f(6) = 6, \dots, f(15) = 189, f(16) = 277$. Thus the number of sequences required is 277.

3. Let $R$ denote the specified region, i.e. $\{(x,y) \mid x^4+y^4 \le x^2-x^2y^2+y^2\}$. Then $R$ can be described as the region inside the curve $x^4+x^2y^2+y^4 = x^2+y^2$ ($(x,y) \neq (0,0)$). This can be rewritten as

$$(x^2+y^2-xy)(x^2+y^2+xy) = x^2+y^2.$$

Now change to polar coordinates: write $x = r\cos\theta$, $y = r\sin\theta$; then the equation becomes $(r^2-r^2\cos\theta\sin\theta)(r^2+r^2\cos\theta\sin\theta) = r^2$. Since $r \neq 0$ and $2\cos\theta\sin\theta = \sin 2\theta$, we now have $r^2(1-\frac{1}{4}\sin^2 2\theta) = 1$. Therefore the area $A$ of $R$ is

$$\iint_R r\,dr\,d\theta = \int_0^{2\pi}\int_0^{(1-\frac{1}{4}\sin^2 2\theta)^{-1/2}} r\,dr\,d\theta = \int_0^{2\pi} \frac{d\theta}{2(1-\frac{1}{4}\sin^2 2\theta)}$$

$$= \int_0^{\pi/4} \frac{16\,d\theta}{3+\cos^2 2\theta} = \int_0^{\pi/4} \frac{16\sec^2 2\theta\,d\theta}{4+3\tan^2 2\theta}.$$

Now make the substitution $2z = \sqrt{3}\tan 2\theta$, so $dz = \sqrt{3}\sec^2 2\theta\,d\theta$ and we obtain

$$A = \frac{4}{\sqrt{3}}\int_0^{\infty} \frac{dz}{1+z^2} = 2\pi/\sqrt{3}.$$

4. Ceva's theorem applied to the triangle $ABC$ shows that $\dfrac{AP}{PB}\dfrac{BM}{MC}\dfrac{CN}{NA} = 1$. Since $BM = MC$, we see that $\dfrac{AP}{PB} = \dfrac{AN}{NC}$ and we deduce that $PN$ is parallel to $BC$. Therefore $\angle NPX = \angle PCB = \angle NAX$ and we conclude that $APXN$ is a cyclic quadrilateral. Since that opposite angles of a cyclic quadrilateral sum to $180^\circ$, we see that $\angle APX + \angle XNA = 180^\circ$, and the result follows.

5. Let $\mathscr{T} = \{a_n \mid n \in \mathbb{N}\}$ and for $t$ a positive number, let $A_t = \{n \in \mathbb{N} \mid a_n \ge t\}$. Since $\sum a_n = 1$ and $a_n \ge 0$ for all $n$, we that if $\delta > 0$, then there are only finitely many numbers in $\mathscr{T}$ greater than $\delta$, and also $A_t$ is finite. Thus we may label the nonzero elements of $\mathscr{T}$ as $t_1, t_2, t_3, \ldots$, where $t_1 > t_2 > t_3 > \cdots > 0$. We shall use the notation $X \triangle Y$ to indicate the symmetric difference $\{X \setminus Y \cup Y \setminus X\}$ of two subsets $X, Y$.

Consider the sum

$$\sum_{i\ge 1} (t_i - t_{i+1})|A_{t_i} \triangle \pi^{-1}A_{t_i}|.$$

Note that $n \in A_t \setminus \pi^{-1}A_t$ if and only if $a_n \ge t > a_{\pi n}$, and $n \in \pi^{-1}A_t \setminus A_t$ if and only if $a_n < t \le a_{\pi n}$. Write $a_n = t_p$ and $a_{\pi n} = t_q$. We have three cases to examine:

(a) $a_n = a_{\pi n}$. Then $n$ does not appear in the above sum.

(b) $a_n > a_{\pi n}$. Then $p < q$ and $n$ is in $A_{t_r} \setminus \pi^{-1} A_{t_r}$ whenever $t_p \geq t_r > t_q$, that is $q > r \geq p$ and we get a contribution $(t_p - t_{p+1}) + (t_{p+1} - t_{p+2}) + \cdots + (t_{q-1} - t_q) = t_p - t_q = a_n - a_{\pi n} = |a_n - a_{\pi n}|$.

(c) $a_n < a_{\pi n}$. Then $p > q$ and $n$ is in $\pi^{-1} A_{t_r} \setminus A_{t_r}$ whenever $t_q \geq t_r > t_p$, that is $p > r \geq q$ and we get a contribution $(t_q - t_{q+1}) + (t_{q+1} - t_{q+2}) + \cdots + (t_{p-1} - t_p) = t_q - t_p = a_{\pi n} - a_n = |a_n - a_{\pi n}|$.

We conclude that

$$\sum_{n=1}^{\infty} |a_n - a_{\pi n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \pi A_{t_i}|,$$

because $|A_{t_i} \triangle \pi^{-1} A_{t_i}| = |A_{t_i} \triangle \pi A_{t_i}|$. Similarly

$$\sum_{n=1}^{\infty} |a_n - a_{\rho n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \rho A_{t_i}|,$$

and we deduce that

$$\sum_{n=1}^{\infty} (|a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}|) = \sum_{i \geq 1} (t_i - t_{i+1})(|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|).$$

Therefore $\sum_{i \geq 1} (t_i - t_{i+1})(|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|) < \varepsilon$. We also have $\sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i}| = 1$. Therefore for some $i$, we must have $|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}| < \varepsilon |A_{t_i}|$ and the result follows.

6. Multiply $a^4 - 3a^2 + 1$ by $b$ and subtract $(a^3 - 3a)(ab - 1)$ to obtain $a^3 - 3a + b$. Now multiply by $b$ and subtract $a^2(ab - 1)$ to obtain $a^2 - 3ab + b^2$. Thus we want to know when $ab - 1$ divides $(a - b)^2 - 1$, where $a, b$ are positive integers. We cannot have $a = b$, because $a^2 - 1$ does not divide $-1$. We now assume that $a > b$.

   Suppose $ab - 1$ does divide $(a - b)^2 - 1$ where $a, b$ are positive integers. Write $(a - b)^2 - 1 = k(ab - 1)$, where $k$ is an integer. Since $(a - b)^2 - 1 \geq 0$, we see that $k$ is nonnegative. If $k = 0$, then we have $(a - b)^2 = 1$, so $a - b = \pm 1$. In this case, $ab - 1$ does divide $a^4 - 3a^2 + 1$, because $a^4 - 3a^2 + 1 = (a^2 + a - 1)(a^2 - a - 1)$. We now assume that $k \geq 1$.

   Now fix $k$ and choose $a, b$ with $b$ as small as possible. Then we have $a^2 + a(-2b - kb) + b^2 + k - 1 = 0$. Consider the quadratic equation $x^2 + x(-2b -$

$kb) + b^2 + k - 1 = 0$. This has an integer root $x = a$. Let $v$ be its other root. Since the sum of the roots is $2b + kb$, we see that $v$ is also an integer. Also $av = b^2 + k - 1$. Since $b, k \geq 1$, we see that $v$ is also positive. We want to show that $v < b$; if this was not the case, then we would have $b^2 + k - 1 \geq ab$, that is $k \geq ab - b^2 + 1$. We now obtain

$$(a-b)^2 - 1 \geq (ab - b^2 + 1)(ab - 1).$$

This simplifies to $a^2 - ab \geq (ab - b^2 + 1)ab$, that is $a - b \geq (ab - b^2 + 1)b$ and we obtain $(a-b)(b^2 - 1) + b \leq 0$, which is not the case. Thus $v < b$ and we have $v^2 + v(-2b - k) + b^2 + k - 1 = 0$. Set $u = b$. Then we have $(u - v)^2 - 1 = k(uv - 1)$, where $u, v$ are positive and $v < b$. By minimality of $b$, we conclude that there are no $a, b$ such that $(a-b)^2 - 1 = k(ab - 1)$.

Putting this altogether, the positive integers required are all $a, b$ such that $b = a \pm 1$.

7. Note that for fixed $x > 1$, the sequence $1/f_n(x)$ is decreasing with respect to $n$ and positive, so the given limit exists which means that $g$ is well-defined. Next we show that $g(e^{1/e}) \geq 1/e$, equivalently $\lim_{n\to\infty} f_n(e^{1/e}) \leq e$. To do this, we show by induction that $f_n(e^{1/e}) \leq e$ for all positive integers $n$. Certainly $f_1(e^{1/e}) = e^{1/e} \leq e$. Now if $f_n(e^{1/e}) \leq e$, then

$$f_{n+1}(e^{1/e}) = (e^{1/e})^{f_n(e^{1/e})} \leq (e^{1/e})^e = e,$$

so the induction step passes and we have proven that $g(e^{1/e}) \geq 1/e$.

We now prove that $g(x) = 0$ for all $x > e^{1/e}$; this will show that $g$ is discontinuous at $x = e^{1/e}$. We need to prove that $\lim_{n\to\infty} f_n(x) = \infty$. If this is not the case, then we may write $\lim_{n\to\infty} f_n(x) = y$ where $y$ is a positive number $> 1$. We now have

$$y = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_{n+1}(x) = x^{\lim_{n\to\infty} f_n(x)} = x^y.$$

Therefore $\ln y = y \ln x$ and $x = y^{1/y}$. Since $(dx/dy)/x = (1 - \ln y)/y^2$, we see by considering the graph of $y^{1/y}$ that it reaches its maximum when $y = e$, and we deduce that $x \leq e^{1/e}$. This is a contradiction and we conclude that $\lim_{n\to\infty} f_n(x) = 0$. Thus we have shown that $g(x)$ is discontinuous at $x = e^{1/e}$.