30th VTRMC, 2008, Solutions

1. Write \( f(x) = xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x \). First we look for local maxima, so we need to solve \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \). Now \( \frac{\partial f}{\partial x} = y^3 + 3x^2z - z^3 - 3x^2y \). If \( y = z \), then \( f(x, y, z) = 0 \) and this is not a maximum. Thus we may divide by \( y - z \) and then \( \frac{\partial f}{\partial x} = 0 \) yields \( y^2 + yz + z^2 = 3x^2 \). Similarly \( x^2 + xz + z^2 = 3y^2 \) and \( x^2 + xy + y^2 = 3z^2 \). Adding these three equations, we obtain \((x-y)^2 + (y-z)^2 + (z-x)^2 = 0\), which yields \( x = y = z \). This does not give a maximum, because \( f = 0 \) in this case, and we conclude that the maximum of \( f \) must occur on the boundary of the region, so at least one of \( x, y, z \) is 0 or 1.

Let’s look at \( f \) on the side \( x = 0 \). Here \( f = yz^3 - y^3z \) and \( 0 \leq y \leq 1, 0 \leq z \leq 1 \). To find local maxima, we solve \( \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \). This yields \( y = z = 0 \) and \( f = 0 \), which is not a maximum, so the maximum occurs on the edges of the region considered. If \( y \) or \( z = 0 \), we get \( f = 0 \) which is not a maximum. If \( y = 1 \), then \( f = z^3 - z \leq 0 \), which won’t give a maximum. Finally if \( z = 1 \), then \( f = y - y^3 \). Since \( df/dy = 1 - 3y^2 \), we see that \( f \) has a maximum at \( y = 1/\sqrt{3} \). This gives that the maximum value of \( f \) on \( x = 0 \) is \( 1/\sqrt{3} - 1/\sqrt{3}^3 = 2\sqrt{3}/9 \).

Similarly if \( y \) or \( z = 0 \), the maximum value of \( f \) is \( 2\sqrt{3}/9 \). Now let’s look at \( f \) on the side \( x = 1 \). Here \( f = y^3 + yz^3 + z - y^3z - z^3 \). Again we first look for local maxima: \( \frac{\partial f}{\partial y} = 3y^2 + z^3 - 1 - 3y^2z \). Then \( \frac{\partial f}{\partial y} = 0 \) yields either \( z = 1 \) or \( 3y^2 = z^2 + z + 1 \). If \( z = 1 \), then \( f = 0 \) which is not a maximum, so \( 3y^2 = z^2 + z + 1 \). Similarly \( 3z^2 = y^2 + y + 1 \). Adding these two equations, we find that \( y^2 - y/2 + z^2 - z/2 = 1 \). Thus \( (y - 1/2)^2 + (z - 1/2)^2 = 3/2 \). This has no solution in the region considered \( 0 \leq y \leq 1, 0 \leq z \leq 1 \). Thus \( f \) must have a maximum on one of the edges. If \( y \) or \( z \) is 0, then we are back in the previous case. On the other hand if \( y \) or \( z \) is 1, then \( f = 0 \), which is not a maximum.

We conclude that the maximum value of \( f \) on \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \) is \( 2\sqrt{3}/9 \).

2. For each positive integer \( n \), let \( f(n) \) denote the number of sequences of 1’s and 3’s that sum to \( n \). Then \( f(n+3) = f(n+2) + f(n) \), and we have \( f(1) = 1, f(2) = 1, \) and \( f(3) = 2 \). Thus \( f(4) = f(3) + f(1) = 3, f(5) = f(4) + f(2) = 4, f(6) = 6, \ldots, f(15) = 189, f(16) = 277 \). Thus the number of sequences required is 277.
3. Let \( R \) denote the specified region, i.e. \( \{(x, y) \mid x^4 + y^4 \leq x^2 - x^2y^2 + y^2\} \). Then \( R \) can be described as the region inside the curve \( x^4 + x^2y^2 + y^4 = x^2 + y^2 \ ( (x, y) \neq (0, 0)) \). This can be rewritten as
\[
(x^2 + y^2 - xy)(x^2 + y^2 + xy) = x^2 + y^2.
\]
Now change to polar coordinates: write \( x = r \cos \theta, \ y = r \sin \theta \); then the equation becomes \( (r^2 - r^2 \cos \theta \sin \theta)(r^2 + r^2 \cos \theta \sin \theta) = r^2 \). Since \( r \neq 0 \) and \( 2 \cos \theta \sin \theta = \sin 2\theta \), we now have \( r^2(1 - \frac{1}{2} \sin^2 2\theta) = 1 \). Therefore the area \( A \) of \( R \) is
\[
\int_R r \, dr \, d\theta = \int_0^{2\pi} \int_0^{(1 - \frac{1}{2} \sin^2 2\theta)^{-1/2}} r \, dr \, d\theta = \int_0^{2\pi} \frac{d\theta}{2(1 - \frac{1}{4} \sin^2 2\theta)}
\]
\[
= \int_0^{\pi/4} \frac{16 \, d\theta}{3 + \cos^2 2\theta} = \int_0^{\pi/4} \frac{16 \sec^2 2\theta \, d\theta}{4 + 3 \tan^2 2\theta}.
\]
Now make the substitution \( 2z = \sqrt{3} \tan 2\theta \), so \( dz = \sqrt{3} \sec^2 2\theta \, d\theta \) and we obtain
\[
A = \frac{4}{\sqrt{3}} \int_0^\infty \frac{dz}{1 + z^2} = \frac{2\pi}{\sqrt{3}}.
\]

4. Ceva’s theorem applied to the triangle \( ABC \) shows that \( \frac{AP}{PB} \cdot \frac{BM}{MC} \cdot \frac{CN}{NA} = 1 \).

Since \( BM = MC \), we see that \( \frac{AP}{PB} = \frac{AN}{NC} \) and we deduce that \( PN \) is parallel to \( BC \). Therefore \( \angle NPX = \angle PCB = \angle NAX \) and we conclude that \( APXN \) is a cyclic quadrilateral. Since that opposite angles of a cyclic quadrilateral sum to 180°, we see that \( \angle APX + \angle XNA = 180° \), and the result follows.

5. Let \( \mathcal{T} = \{a_n \mid n \in \mathbb{N}\} \) and for \( t \) a positive number, let \( A_t = \{n \in \mathbb{N} \mid a_n \geq t\} \).
Since \( \sum a_n = 1 \) and \( a_n \geq 0 \) for all \( n \), we that if \( \delta > 0 \), then there are only finitely many numbers in \( \mathcal{T} \) greater than \( \delta \), and also \( A_t \) is finite. Thus we may label the nonzero elements of \( \mathcal{T} \) as \( t_1, t_2, t_3, \ldots \), where \( t_1 > t_2 > t_3 > \cdots > 0 \). We shall use the notation \( X \triangle Y \) to indicate the symmetric difference \( \{X \setminus Y \cup Y \setminus X\} \) of two subsets \( X, Y \).

Consider the sum
\[
\sum_{i \geq 1} (t_i - t_{i+1}) \mid A_{t_i} \triangle \pi^{-1} A_{t_i} \mid.
\]
Note that \( n \in A_t \setminus \pi^{-1} A_t \) if and only if \( a_n \geq t > a_{t_n} \), and \( n \in \pi^{-1} A_t \setminus A_t \) if and only if \( a_n < t \leq a_{t_n} \). Write \( a_n = t_p \) and \( a_{t_n} = t_q \). We have three cases to examine:
(a) $a_n = a_{\pi n}$. Then $n$ does not appear in the above sum.

(b) $a_n > a_{\pi n}$. Then $p < q$ and $n$ is in $A_r \setminus \pi^{-1}A_r$ whenever $t_p \geq t_r > t_q$, that is $q > r \geq p$ and we get a contribution $(t_p - t_{p+1}) + (t_{p+1} - t_{p+2}) + \cdots + (t_{q-1} - t_q) = t_p - t_q = a_n - a_{\pi n} = |a_n - a_{\pi n}|$.

(c) $a_n < a_{\pi n}$. Then $p > q$ and $n$ is in $\pi^{-1}A_r \setminus A_r$ whenever $t_q \geq t_r > t_p$, that is $p > r \geq q$ and we get a contribution $(t_q - t_{q+1}) + (t_{q+1} - t_{q+2}) + \cdots + (t_{p-1} - t_p) = t_q - t_p = a_{\pi n} - a_n = |a_n - a_{\pi n}|$.

We conclude that
\[
\sum_{n=1}^{\infty} |a_n - a_{\pi n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \pi A_{t_i}|,
\]
because $|A_{t_i} \triangle \pi^{-1}A_{t_i}| = |A_{t_i} \triangle \pi A_{t_i}|$. Similarly
\[
\sum_{n=1}^{\infty} |a_n - a_{\rho n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \rho A_{t_i}|,
\]
and we deduce that
\[
\sum_{n=1}^{\infty} (|a_n - a_{\pi n}| + |a_n - a_{\rho n}|) = \sum_{i \geq 1} (t_i - t_{i+1}) (|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|).
\]
Therefore $\sum_{i \geq 1} (t_i - t_{i+1}) (|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|) < \varepsilon$. We also have $\sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i}| = 1$. Therefore for some $i$, we must have $|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}| < \varepsilon |A_{t_i}|$ and the result follows.

6. Multiply $a^4 - 3a^2 + 1$ by $b$ and subtract $(a^3 - 3a)(ab - 1)$ to obtain $a^3 - 3a + b$. Now multiply by $b$ and subtract $a^2(ab - 1)$ to obtain $a^2 - 3ab + b^2$. Thus we want to know when $ab - 1$ divides $(a - b)^2 - 1$, where $a, b$ are positive integers. We cannot have $a = b$, because $a^2 - 1$ does not divide $-1$. We now assume that $a > b$.

Suppose $ab - 1$ does divide $(a - b)^2 - 1$ where $a, b$ are positive integers. Write $(a - b)^2 - 1 = k(ab - 1)$, where $k$ is an integer. Since $(a - b)^2 - 1 \geq 0$, we see that $k$ is nonnegative. If $k = 0$, then we have $(a - b)^2 = 1$, so $a - b = \pm 1$. In this case, $ab - 1$ does divide $a^4 - 3a^2 + 1$, because $a^4 - 3a^2 + 1 = (a^2 + a - 1)(a^2 - a - 1)$. We now assume that $k \geq 1$.

Now fix $k$ and choose $a, b$ with $b$ as small as possible. Then we have $a^2 + a(-2b - kb) + b^2 + k - 1 = 0$. Consider the quadratic equation $x^2 + x(-2b - k) + 1$.
\( kb + b^2 + k - 1 = 0 \). This has an integer root \( x = a \). Let \( v \) be its other root. Since the sum of the roots is \( 2b + kb \), we see that \( v \) is also an integer. Also \( av = b^2 + k - 1 \). Since \( b, k \geq 1 \), we see that \( v \) is also positive. We want to show that \( v < b \); if this was not the case, then we would have \( b^2 + k - 1 \geq ab \), that is \( k \geq ab - b^2 + 1 \). We now obtain

\[(a-b)^2 - 1 \geq (ab - b^2 + 1)(ab-1).
\]

This simplifies to \( a^2 - ab \geq (ab - b^2 + 1)ab \), that is \( a - b \geq (ab - b^2 + 1)b \) and we obtain \( (a-b)(b^2 - 1) + b \leq 0 \), which is not the case. Thus \( v < b \) and we have \( v^2 + v(-2b-k) + b^2 + k - 1 = 0 \). Set \( u = b \). Then we have \((u-v)^2 - 1 = k(uv - 1)\), where \( u, v \) are positive and \( v < b \). By minimality of \( b \), we conclude that there are no \( a, b \) such that \((a-b)^2 - 1 = k(ab-1)\).

Putting this altogether, the positive integers required are all \( a, b \) such that \( b = a \pm 1 \).

7. Note that for fixed \( x > 1 \), the sequence \( 1/f_n(x) \) is decreasing with respect to \( n \) and positive, so the given limit exists which means that \( g \) is well-defined. Next we show that \( g(e^{1/e}) \geq 1/e \), equivalently \( \lim_{n \to \infty} f_n(e^{1/e}) \leq e \). To do this, we show by induction that \( f_n(e^{1/e}) \leq e \) for all positive integers \( n \). Certainly \( f_1(e^{1/e}) = e^{1/e} \leq e \). Now if \( f_n(e^{1/e}) \leq e \), then

\[ f_{n+1}(e^{1/e}) = (e^{1/e})f_n(e^{1/e}) \leq (e^{1/e})e = e, \]

so the induction step passes and we have proven that \( g(e^{1/e}) \geq 1/e \).

We now prove that \( g(x) = 0 \) for all \( x > e^{1/e} \); this will show that \( g \) is discontinuous at \( x = e^{1/e} \). We need to prove that \( \lim_{n \to \infty} f_n(x) = \infty \). If this is not the case, then we may write \( \lim_{n \to \infty} f_n(x) = y \) where \( y \) is a positive number \( > 1 \). We now have

\[ y = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_{n+1}(x) = x^{\lim_{n \to \infty} f_n(x)} = x^y. \]

Therefore \( \ln y = y \ln x \) and \( x = y^{1/y} \). Since \( (dx/dy)/x = (1 - \ln y)/y^2 \), we see by considering the graph of \( y^{1/y} \) that it reaches its maximum when \( y = e \), and we deduce that \( x \leq e^{1/e} \). This is a contradiction and we conclude that \( \lim_{n \to \infty} f_n(x) = 0 \). Thus we have shown that \( g(x) \) is discontinuous at \( x = e^{1/e} \).