29th VTRMC, 2007, Solutions

1. Let $I = \int \frac{d\theta}{2+\tan\theta}$ We make the substitution $y = \tan\theta$. Then $dy = \sec^2\theta d\theta = (1+y^2)d\theta$ and we find that
$$I = \int \frac{dy}{(1+y^2)(2+y)}.$$
Since $5/((1+y^2)(2+y)) = 1/(2+y) - y/(1+y^2) + 2/(1+y^2)$, we find that
$$5I = \int \frac{dy}{y+2} - \int \frac{ydy}{1+y^2} + \int \frac{2dy}{1+y^2} = \ln(2+y) - (\ln(1+y^2))/2 + 2\tan^{-1} y.$$
Therefore $5I = \ln\frac{2+\tan\theta}{\sec\theta} + 2\theta = \ln(2\cos\theta + \sin\theta) + 2\theta$ and we deduce that
$$I = \frac{2\theta + \ln(2\cos\theta + \sin\theta)}{5},$$
$$\text{hence} \quad \int_0^x \frac{d\theta}{2+\tan\theta} = \frac{2x + \ln(2\cos x + \sin x) - \ln 2}{5}.$$
Plugging in $x = \pi/4$, we conclude that
$$\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi + 2\ln(3/\sqrt{2}) - 2\ln 2}{10} = \frac{\pi + \ln(9/8)}{10}.$$

2. Let $A = 1 + \sum_{n=1}^{\infty}(n+1)/(2n+1)!$ and $B = \sum_{n=1}^{\infty} n/(2n+1)!$, so A and B are the values of the sums in (a) and (b) respectively. Now
$$A + B = \sum_{n=0}^{\infty} 1/(2n)! = (e + e^{-1})/2,$$
$$A - B = \sum_{n=0}^{\infty} 1/(2n+1)! = (e - e^{-1})/2.$$
Therefore $A = e/2$ and $B = 1/(2e)$.

3. We make the substitution $y = e^u$ where u is a function of x to be determined. Then $y' = u'e^u$ and plugging into the given differential equation, we find that $u'e^u = e^u u + e^u e^x$, hence $u' - u = e^x$. This is a first order linear differential equation which can be solved in several ways, for example one method would be to multiply by the integrating factor e^{-x}. We obtain the general solution $u = xe^x + Ce^x$, where C is an arbitrary constant. We are given $y = 1$ when $x = 0$, and then $u = 0$. Therefore $u = xe^x$ and we conclude that $y = e^{xe^x}$.

4. Ceva's theorem applied to the triangle ABC shows that $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$. Since RP bisects $\angle BRC$, we see that $\frac{BP}{PC} = \frac{BR}{RC}$. Therefore $\frac{AR}{RC} = \frac{AQ}{QC}$, consequently $\angle ARQ = \angle QRC$ and the result follows.

5. Let

$$\begin{aligned}
A &= (2+\sqrt{5})^{100}((1+\sqrt{2})^{100} + (1+\sqrt{2})^{-100})\\
B &= (\sqrt{5}-2)^{100}((1+\sqrt{2})^{100} + (1+\sqrt{2})^{-100})\\
C &= (\sqrt{5}+2)^{100} + (\sqrt{5}-2)^{100}\\
D &= (\sqrt{2}+1)^{100} + (\sqrt{2}-1)^{100}
\end{aligned}$$

First note that C and D are integers; one way to see this is to use the binomial theorem. Also $\sqrt{2}-1 = (\sqrt{2}+1)^{-1}$. Thus $A+B = CD$ is an integer. Now $\sqrt{5}-2 < 1/4$, $\sqrt{2}+1 < 2.5$ and $\sqrt{2}-1 < 1$. Therefore $0 < B < (5/8)^{100} + (1/4)^{100} < 10^{-4}$. We conclude that there is a positive number $\varepsilon < 10^{-4}$ such that $A+\varepsilon$ is an integer, hence the third digit after the decimal point of the given expression A is 9.

6. Suppose $\det(A^2+B^2) = 0$. Then A^2+B^2 is not invertible and hence there exists a nonzero $n \times 1$ matrix (column vector) u with real entries such that $(A^2+B^2)u = 0$. Then $u'A^2u + u'B^2u = 0$, where u' denotes the transpose of u, a $1 \times n$ matrix. Therefore $(Au)'(Au) + (Bu)'(Bu) = 0$ and we deduce that $u'A = u'B = 0$, consequently $u'(AX+BY) = 0$. This shows that $\det(AX+BY) = 0$, a contradiction and the result follows.

7. We claim that $x^{1/(\ln(\ln x))^2} > (\ln x)^2$ for large x. Indeed by taking logs, we need $(\ln x)/(\ln(\ln x))^2 > 2\ln(\ln(x))$, that is $\ln x > 2(\ln(\ln x))^3$. So by making the substitution $y = \ln x$, we want $y > 2(\ln y)^3$, which is true for y large. It now follows that for large n,

$$n^{-(1+\frac{1}{(\ln(\ln n))^2})} = \frac{1}{n}\frac{1}{n^{1/(\ln(\ln n))^2}} < \frac{1}{n(\ln n)^2}.$$

However $\sum 1/(n(\ln n)^2)$ is well known to be convergent, by using the integral test, and it now follows from the basic comparison test that the given series is also convergent.