25th VTRMC, 2003, Solutions

1. The probability of p gains is the coefficient of $(1/2)^p(1/2)^{n-p}$ in $(1/2+1/2)^n$. Therefore, without the insider trading scenario, on average the investor will have $10000(3/5+9/20)^n$ dollars at the end of n days. With the insider trading, the first term $(3/5)^n$ becomes 0. Therefore on average the investor will have

$$10000\left(\frac{21}{20}\right)^n - 10000\left(\frac{3}{5}\right)^n$$

dollars at the end of n days.

2. We have

$$-\ln(1-x) = x + x^2/2 + x^3/3 + \cdots = \sum_{i=1}^{\infty} \frac{x^n}{n}.$$

Therefore

$$\begin{aligned}(1-x)\ln(1-x) &= -x + x^2/2 + x^3/6 + \ldots \\ &= -x + \sum_{i=1}^{\infty} x^{n+1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = -x + \sum_{i=1}^{\infty} \frac{x^{n+1}}{n(n+1)}.\end{aligned}$$

Dividing by x, we deduce that

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = 1 + \frac{1-x}{x}\ln(1-x)$$

for $x \neq 0$, and the sum is 0 for $x = 0$.

3. Let I denote the 2 by 2 identity matrix. Since $A = A^{-1}$, we see that $A^2 = I$ and hence the eigenvalues λ of A must satisfy $\lambda^2 = 1$, so $\lambda = \pm 1$. First consider the case $\det A = 1$. Then A has a repeated eigenvalue ± 1, and A is similar to $\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}$ where $r = \pm 1$ and $s = 0$ or 1. Since $A^2 = I$, we see that $s = 0$ and we conclude that $A = \pm I$.

   Now suppose $\det A = -1$. Then the eigenvalues of A must be $1, -1$, so the trace of A must be 0, which means that A has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where

a, b are complex numbers satisfying $a^2 + b^2 = 1$. Therefore $b = (1 - a^2)^{1/2}$ (where the exponent 1/2 means one of the two complex numbers whose square is $1 - a^2$). We conclude that the matrices satisfying $A = A' = A^{-1}$ are $\pm I$ and $\begin{pmatrix} a & (1-a^2)^{1/2} \\ (1-a^2)^{1/2} & -a \end{pmatrix}$ where a is any complex number.

4. Set $R = e^{2\pi i/7} = \cos 2\pi/7 + i \sin 2\pi/7$. Since $R \neq 1$ and $R^7 = 1$, we see that $1 + R + \cdots + R^6 = 0$. Now for n an integer, $R^n = \cos 2n\pi/7 + i \sin 2n\pi/7$. Thus by taking the real parts and using $\cos(2\pi - x) = \cos x$, $\cos(\pi - x) = -\cos x$, we obtain

$$1 + 2\cos\frac{2\pi}{7} - 2\cos\frac{\pi}{7} - 2\cos\frac{3\pi}{7} = 0.$$

Since $\cos\pi/7 + \cos 3\pi/7 = 2\cos(2\pi/7)(\cos\pi/7)$, the above becomes

$$4\cos\frac{2\pi}{7}\cos\frac{\pi}{7} - 2\cos\frac{2\pi}{7} = -1.$$

Finally $\cos(2\pi/7) = 2\cos^2(\pi/7) - 1$, hence $(2\cos^2(\pi/7) - 1)(4\cos(\pi/7) - 2) = -1$ and we conclude that $8\cos^3(\pi/7) - 4\cos^2(\pi/7) - 4\cos(\pi/7) = -1$. Therefore the rational number required is $-1/4$.

5. Since $\angle ABC + \angle PQC = 90$ and $\angle ACB + \angle PRB = 90$, we see that $\angle QPR = \angle ABC + \angle ACB$. Now X, Y, Z being the midpoints of BC, CA, AB respectively tells us that AY is parallel to ZX, AZ is parallel to XY, and BX is parallel to YZ. We deduce that $\angle ZXY = \angle BAC$ and hence $\angle QPR + \angle ZXY = 180$. Therefore the points P, Z, X, Y lie on a circle and we deduce that $\angle QPX = \angle ZYX$. Using BZ parallel to XY and BX parallel to ZY from above, we conclude that $\angle ZYX = \angle ABC$. Therefore $\angle QPX + \angle PQX = \angle ABC + \angle PQX = 90$ and the result follows.

6. Set $g = f^2$. Note that g is continuous, $g^3(x) = x$ for all x, and $f(x) = x$ for all x if and only if $g(x) = x$ for all x. Suppose $y \in [0, 1]$ and $f(y) \neq y$. Then the numbers $y, f(y), f^2(y)$ are distinct. Replacing y with $f(y)$ or $f^2(y)$ and f with g if necessary, we may assume that $y < f(y) < f^2(y)$. Choose $a \in (f(y), f^2(y))$. Since f is continuous, there exists $p \in (y, f(y))$ and $q \in (f(y), f^2(y))$ such that $f(p) = a = f(q)$. Thus $f(p) = f(q)$, hence $f^3(p) = f^3(q)$ and we deduce that $p = q$. This is a contradiction because $p < f(y) < q$, and the result follows.

7. Let the tetrahedron have vertices A,B,C,D and let X denote the midpoint of BC. Then $AX = \sqrt{1-1/4} = \sqrt{3}/2$ and we see that ABC has area $\sqrt{3}/4$. Let R,S,T,U denote the regions vertically above and distance at most 1 from ABC, BCD, ABD, ACD respectively. Then the volumes of R,S,T and U are all $\sqrt{3}/4$. Since these regions are disjoint, they will contribute $\sqrt{3}$ to the volume required.

   Let Y denote the point on AX which is vertically below D. Then Y is the center of ABC (i.e. where the medians meet), in particular $\angle YBX = \pi/6$ and we see that $BY = 1/\sqrt{3}$. Therefore $DY = \sqrt{1-1/3} = \sqrt{2/3}$ and we deduce that $ABCD$ has volume

$$\frac{1}{3} * \frac{\sqrt{3}}{4} * \sqrt{2/3} = \frac{\sqrt{2}}{12}.$$

   Next consider the region which is distance 1 from BC and is between R and S. We need the angle between R and S, and for this we find the angle between DX and DY. Now $DX = AX = \sqrt{3}/2$ and $DY = \sqrt{2/3}$. Therefore $XY = \sqrt{3/4-2/3} = 1/(2\sqrt{3})$. If $\theta = \angle YDX$, then $\sin\theta = XY/DX = 1/3$. We deduce that the angle between R and S is $\pi/2 + \theta = \pi/2 + \sin^{-1} 1/3$. Therefore the region at distance 1 from BC and between R and S has volume $\pi/4 + (\sin^{-1} 1/3)/2$. There are 6 such regions, which contribute $3\pi/2 + 3\sin^{-1} 1/3$ to the volume required.

   For the remaining volume, we shrink the sides of the tetrahedron to zero. This keeps the remaining volume constant, but the volumes above go to zero. We are left with the volume which is distance 1 from the center of the pyramid, which is $4\pi/3$. Since $3\pi/2 + 4\pi/3 = 17\pi/6$, we conclude that the volume of the region consisting of points which are distance at most 1 from $ABCD$ is $\sqrt{3} + \sqrt{2}/12 + 17\pi/6 + 3\sin^{-1}(1/3) \approx 11.77$. Other expressions for this are $\sqrt{3} + \sqrt{2}/12 + 13\pi/3 - 3\cos^{-1}(1/3)$ and $\sqrt{3} + \sqrt{2}/12 + 13\pi/3 - 6\sin^{-1}(1/\sqrt{3})$.