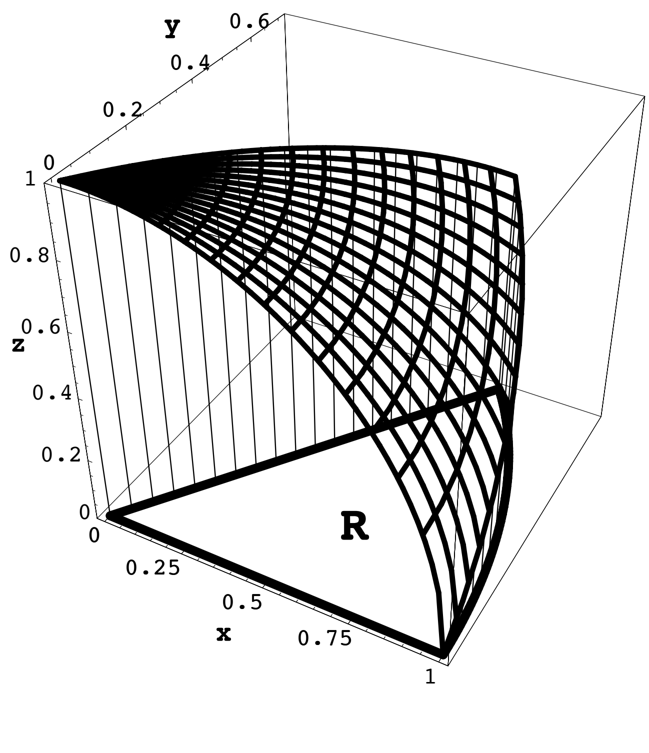
23rd VTRMC, 2001, Solutions

1. We calculate the volume of the region which is in the first octant and above $\{(x,y,0) \mid x \geq y\}$; this is 1/16 of the required volume. The volume is above R, where R is the region in the xy-plane and bounded by $y=0$, $y=x$ and $y=\sqrt{1-x^2}$, and below $z=\sqrt{1-x^2}$. This volume is

$$\int_0^{1/\sqrt{2}}\int_0^x\int_0^{\sqrt{1-x^2}} dzdydx + \int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx$$
$$= \int_0^{1/\sqrt{2}}\int_0^x \sqrt{1-x^2}\,dydx + \int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dydx$$
$$= \int_0^{1/\sqrt{2}} x\sqrt{1-x^2}\,dx + \int_{1/\sqrt{2}}^1 (1-x^2)\,dx$$
$$= [-(1-x^2)^{3/2}/3]_0^{1/\sqrt{2}} + [x-x^3/3]_{1/\sqrt{2}}^1$$
$$= \frac{1}{3} - \frac{1}{6\sqrt{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}} - \frac{1}{\sqrt{2}}$$
$$= 1 - 1/\sqrt{2}.$$

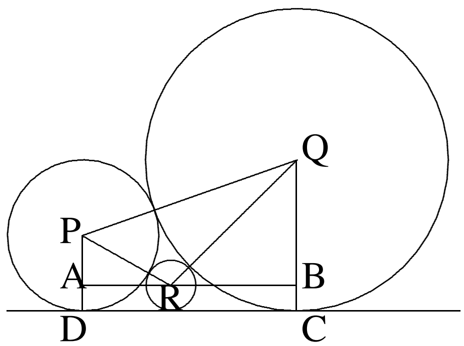
Therefore the required volume is $16-8\sqrt{2}$.



2. Let the circle with radius 1 have center P, the circle with radius 2 have center Q, and let R be the center of the third circle, as shown below. Let $\overline{AR} = a$, $\overline{RB} = b$, and let the radius of the third circle be r. By Pythagoras on the triangles PAR and QRB, we obtain

$$(1-r)^2 + a^2 = (1+r)^2, \quad (2-r)^2 + b^2 = (2+r)^2.$$

Therefore $a^2 = 4r$ and $b^2 = 8r$. Also $(a+b)^2 + 1 = 9$, so $2\sqrt{r} + 2\sqrt{2r} = \sqrt{8}$ and we deduce that $r = 6 - 4\sqrt{2}$.



3. Let m, n be a positive integers where $m \leq n$. For each $m \times m$ square in an $n \times n$ grid, replace it with the $(m-1) \times (m-1)$ square obtained by deleting the first row and column; this means that the 1×1 squares become nothing. Then these new squares (don't include the squares which are nothing) are in a one-to-one correspondence with the squares of the $(n-1) \times (n-1)$ grid obtained by deleting the first row and column of the $n \times n$ square. Therefore S_{n-1} is the number of squares in an $n \times n$ grid which have size at least 2×2. Since there are n^2 1×1 squares in an $n \times n$ grid, we deduce that $S_n = S_{n-1} + n^2$.

   Thus

$$S_8 = 1^2 + 2^2 + \cdots + 8^2 = 204.$$

4. If $p \leq q < (p+1)^2$, then p divides q if and only if $q = p^2, p^2 + p$ or $p^2 + 2p$. Therefore if $a_k = p^2$, then $a_{k+3} = (p+1)^2$. Since $a_1 = 1^2$, we see that

$$a_{10000} = (1 + 9999/3)^2 = 3334^2 = 11115556.$$

5. Let $a_n = n^n x^n / n!$. First we use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} x^{n+1} n!}{n^n x^n (n+1)!} = (1 + 1/n)^n x.$$

Since $\lim_{n\to\infty}(1+1/n)^n = e$, we see that the interval of convergence is of the form $\{-1/e, 1/e\}$, where we need to decide whether the interval is open or closed at its two endpoints. By considering $\int_n^{n+1} dx/x$, we see that

$$\frac{1}{n+1} < \ln(1+1/n) < \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right) < \frac{3}{3n+1}$$

because $1/n$ is a concave function, consequently $(1+1/n)^{n+1/3} < e < (1+1/n)^{n+1}$. Therefore when $|x| = 1/e$, we see that $|a_n|$ is a decreasing sequence. Furthermore by induction on n and the left hand side of the last inequality, we see that $|a_n| < 1/\sqrt[3]{n}$. Thus when $x = -1/e$, we see that $\lim_{n\to\infty} a_n = 0$ and it follows that the given series is convergent, by the alternating series test. On the other hand when $x = 1/e$, the series is $\sum n^n/(e^n n!)$. By induction on n and the above inequality, we see that $n^n/(e^n/n!) > 1/(en)$ for all $n > 1$. Since $\sum 1/n$ is divergent, we deduce that the given series is divergent when $x = 1/e$, consequently the interval of convergence is $[-1/e, 1/e)$.

6. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. If we can find a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $B^2 = A$ or even $4A$ and set $f(x) = \dfrac{ax+b}{cx+d}$, then $f(f(x)) = \dfrac{3x+1}{x+3}$. So we want to find a square root of A. The eigenvalues of A are 2 and 4, and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively. Thus if $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $XAX^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Set $C = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{pmatrix}$. Then $C^2 = XAX^{-1}$, so if we let $B = (X^{-1}CX)^2$, then $B^2 = A$. Since

$$2B = \begin{pmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 2-\sqrt{2} & 2+\sqrt{2} \end{pmatrix}$$

we may define (multiply top and bottom by $1+\sqrt{2}/2$)

$$f(x) = \frac{(3+2\sqrt{2})x+1}{x+3+2\sqrt{2}}.$$

Finally we should remark that f still maps $\mathbb{R}^+$ to $\mathbb{R}^+$. Of course there are many other solutions and answers.

7. Choose $x \in A$ and $y \in B$ so that $f(xy)$ is as large as possible. Suppose we can write xy in another way as ab with $a \in A$ and $b \in B$ (so $a \neq x$). Set $g = ax^{-1}$ and note that $I \neq g \in G$. Therefore either $f(gxy) > f(xy)$ or $f(g^{-1}xy) > f(xy)$. We deduce that either $f(ay) > f(xy)$ or $f(xb) > f(xy)$, a contradiction and the result follows.