22nd VTRMC, 2000, Solutions

1. Let $I = \int_0^\alpha \frac{d\theta}{5-4\cos\theta}$. Using the half angle formula $\cos\theta = 2\cos^2(\theta/2) - 1$, we obtain

$$I = \int_0^\alpha \frac{d\theta}{9-8\cos^2(\theta/2)} = \int_0^\alpha \frac{\sec^2(\theta/2)\,d\theta}{9\sec^2(\theta/2)-8} = \int_0^\alpha \frac{\sec^2(\theta/2)\,d\theta}{9\tan^2(\theta/2)+1}.$$

Now make the substitution $x = 3\tan(\theta/2)$. Then $2\,dx = 3\sec^2(\theta/2)\,d\theta$, consequently

$$3I = \int_0^{3\tan(\alpha/2)} \frac{2\,dx}{1+x^2} = 2\tan^{-1}(3\tan(\alpha/2)).$$

Therefore $I = \frac{2}{3}\tan^{-1}(3\tan(\alpha/2))$. By using the facts that $\tan(\pi/6) = 1/\sqrt{3}$ and $\tan(\pi/3) = \sqrt{3}$, we see that when $\alpha = \pi/3$,

$$I = \frac{2}{3}\tan^{-1}\sqrt{3} = \frac{2\pi}{9}.$$

2. Let J denote the Jordan canonical form of A. Then A and J will have the same trace, and the entries on the main diagonal of J will satisfy $4x^4+1=0$. This equation has roots $\pm 1/2 \pm i/2$, so the trace of A will be a sum of such numbers. But the trace of A is real, hence the imaginary parts must cancel and we see that there must be an even number of terms in the sum. It follows that the trace of A is an integer.

3. Make the substitution $y = x - t$. Then the equation becomes $x' = x^2 - 2xt + 1$. We will show that $\lim_{t\to\infty} x'(t)$ exists and is 0, and then it will follow that $\lim_{t\to\infty} y'(t)$ exists and is -1.

   When $t = 0$, the initial condition tells us that $x = 0$, so $x'(0) = 1$ and we see that $x(t) > 0$ for small t. Suppose for some positive t we have $x(t) \le 0$. Then there is a least positive number T such that $x(T) = 0$. Then $x'(T) = 1$, which leads to a contradiction because $x(t) > 0$ for $t < T$. We deduce that $x(t) > 0$ for all t.

   Now $x' - 1 = x(x-2t)$ and since $x'(0) = 1$, we see that $x - 2t < 0$ for small t. We deduce that for t sufficiently small, $x(t) < t$, consequently $y(t) < 0$ for small t. We now claim that $y(t) < 0$ for all positive t. If this is not the case, then there is a least positive number T such that $y(T) = 0$, and then we must have $y'(S) = 0$ for some S with $0 < S < T$. But from $y' = (y-t)(y+t)$ and $(y+t) > 0$, we would have to have $y(S) = S$, a contradiction because $y(S) < 0$. We deduce that $x(t) < t$ for all positive t.

Now consider $x' = x(x-2t)+1$. Note that we cannot have $x'(t) \geq 0$ for all t, because then $x \to 0$ as $t \to \infty$ which is clearly impossible, consequently x' takes on negative values. Next we have $x'' = 2(xx' - tx' - x)$, so if $x'(t) = 0$, we see that $x''(t) < 0$. We deduce that if $x'(T) < 0$, then $x'(t) < 0$ for all $t > T$. Thus there is a positive number T such that $x'(t) < 0$ for all $t > T$. Now differentiate again to obtain $x''' = 2(xx'' + x'x' - tx'' - 2x')$. Then we see that if $x''(t) = 0$ and $t > T$, then $x'''(t) > 0$, consequently there is a positive number $S > T$ such that either $x''(t) < 0$ for all $t > S$ or $x''(t) > 0$ for all $t > S$. We deduce that $x'(t)$ is monotonic increasing or decreasing for $t > S$ and hence $\lim_{t\to\infty} x'(t)$ exists (possibly infinite).

We now have $x'(t)$ is monotonic and negative for $t > S$, yet $x(t) > 0$ for all $t > S$. We deduce that $\lim_{t\to\infty} x'(t) = 0$ and the result follows.

4. Set $y = \overline{AP}$. Then

$$\begin{aligned} l_2^2 &= (l-x)^2 + y^2 + 2(l-x)y\cos\theta \\ l_1^2 &= x^2 + y^2 - 2xy\cos\theta \end{aligned}$$

Subtracting the second equation from the first we obtain

$$l_2^2 - l_1^2 = l^2 - 2lx + 2ly\cos\theta$$

which yields

$$2y\cos\theta = \frac{l_2^2 - l_1^2 + 2lx - l^2}{l}.$$

From the second equation and the above, we obtain

$$l_1^2 - x^2 + x\frac{l_2^2 - l_1^2 + 2lx - l^2}{l} = y^2.$$

By differentiating the above with respect to x, we now get

$$2x + \frac{l_2^2 - l_1^2 - l^2}{l} = 2y\frac{dy}{dx}$$

and we deduce that $dy/dx = \cos\theta$. Therefore

$$l_2 - l_1 = y(l) - y(0) = \int_0^l \cos\theta \, dx$$

as required.

5. Open out the cylinder so that it is an infinitely long rectangle with width 4. Then the brush paints out two ellipses (one on either side of the cylinder) which have radius $\sqrt{3}$ in the direction of the axis of the cylinder, which we shall call the y-axis, and radius 2 in the perpendicular direction, which we shall call the x-axis. Then the equation of the ellipse is $x^2/4 + y^2/3 = 1$. By considering just one of the ellipses, we see that the area required is

$$4\int_{-1}^{1} y \, dx = 8\sqrt{3}\int_0^1 \sqrt{1 - \frac{x^2}{4}} \, dx.$$

By making the substitution $x = 2\sin\theta$, this evaluates to $6 + \dfrac{4\pi\sqrt{3}}{3}$.

6. Let $\alpha = \sum_{n=1}^{\infty} a_n t^n$. Then

$$\alpha^2 = \sum_{n=1}^{\infty} a_n t^n \sum_{n=1}^{\infty} a_n t^n.$$

Consider the coefficient of t^n on the right hand side of the above; it is

$$a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}$$

for $n \geq 2$ and 0 for $n = 1$. Using the given hypothesis, we see that this is a_n for all $n \geq 2$. We deduce that $t + \alpha^2 = \alpha$. For the problem under consideration, we need to calculate α when $t = 2/9$, so we want to find α when $\alpha^2 - \alpha + 2/9 = 0$. Since the roots of this equation are 1/3 and 2/3, we are nearly finished. However we ought to check that $\alpha \neq 2/3, \infty$.

Suppose this is not the case. Let $\beta_n = \sum_{m=1}^{n} a_m (2/9)^m$. If $\alpha = 2/3$ or ∞, then there exists a positive integer N such that $b_N < 1/3$ and $b_{N+1} \geq 1/3$. Then the same argument as above gives

$$\frac{2}{9} + \beta_N^2 > \beta_{N+1},$$

which is not possible, so the result follows.

7. There are three possible positions for the tiles, namely $\left[{}^{\bullet}{}_{\bullet} \,\middle|\, \bullet\right]$, $\left[\bullet \,\middle|\, {}^{\bullet}{}_{\bullet}\right]$ and $\left[{}^{\bullet}{}_{\bullet} \,\middle|\, {}^{\bullet}{}_{\bullet}\right]$, which we shall call A,B and C respectively. Consider A_{n+2}. Then whatever the $n+1$ st tile is in the chain, we can complete it to a chain of length $n+2$. Therefore $A_{n+2} = A_{n+1} + x$ for some nonnegative integer x. However if the $n+1$ st tile is position A, then there is exactly one way to add a tile to get a chain of length $n+2$ (namely add tile B), whereas if the tile is B or C, then there are exactly two ways to add a tile to get a chain of length $n+2$ (namely add tiles A or C). Therefore x is the number of ways a chain ends of length $n+1$ ends in B plus the number of ways a chain of length end in C, which is precisely A_n. Therefore $A_{n+2} = A_{n+1} + A_n$. This recurrence relation is valid for $n \geq 1$. Since $A_1 = 3$ and $A_2 = 5$, we get $A_3 = 8, A_4 = 13$ and $A_{10} = 233$.