21st VTRMC, 1999, Solutions

1. Since the value of $f(x,y)$ is unchanged when we swap x with y,

$$\int_0^1 \int_0^x f(x+y)\,dydx = \frac{1}{2}\int_0^1 \int_0^1 f(x+y)\,dydx.$$

Also

$$\int_0^1 f(x+y)\,dy = \int_x^{1+x} f(z)\,dz = \int_0^1 f(z)\,dz$$

because $f(z) = f(1+z)$ for all z. Since $\int_0^1 f(z)\,dz = 1999$, we conclude that

$$\int_0^1 \int_0^1 f(x+y)\,dydx = 1999/2.$$

2. For $\alpha = 1, \beta = 0$ and $x = 1$, we have $f(1)f(0) = f(1)$. Therefore $f(0) = 1$. By differentiation

$$\alpha f'(\alpha x) f(\beta x) + \beta f(\alpha x) f'(\beta x) = f'(x)$$

holds for all x, and for all α, β satisfying $\alpha^2 + \beta^2 = 1$. Hence $(\alpha+\beta)f'(0) = f'(0)$ holds. By taking $\alpha = \beta = 1/\sqrt{2}$, we wee that $f'(0) = 0$. Set $c = f''(0)$. By Taylor's theorem, $f(y) = 1 + cy^2/2 + \varepsilon(y^2)$, where $\lim_{y\to 0} \varepsilon(y^2)/y^2 = 0$. By taking $\alpha = \beta = 1/\sqrt{2}$ again, we see that $f(x/\sqrt{2})^2 = f(x)$ for all x. By repetition, for every positive integer m,

$$f(x) = \left(f(2^{-m/2}x)\right)^{2^m}.$$

Now fix any x, and $\delta > 0$. There is a positive integer N such that for all $m \geq N$, $2^{-m}(|c| + \delta)x^2 < 1$, and

$$\left(1 + 2^{-m-1}(c-\delta)x^2\right)^{2^m} \leq f(x) \leq \left(1 + 2^{-m-1}(c+\delta)x^2\right)^{2^m}.$$

Now let $m \to \infty$. We obtain

$$e^{(c-\delta)x^2/2} \leq f(x) \leq e^{(c+\delta)/x^2}.$$

Since δ was arbitrary, $f(x) = e^{cx^2/2}$. Using the condition $f(1) = 2$, we conclude that $f(x) = 2^{x^2}$.

3. Note that any eigenvalue of A_n has absolute value at most M, because the sum of the absolute values of the entries in any row of A_n is at most M. We may assume that $M > 1$. By considering the characteristic polynomial, we see that the product of nonzero eigenvalues of A_n is a nonzero integer. Write $d = e_n(\delta)$. Then we have $M^n\delta^d > 1$. This can be written as

$$\frac{e_n(\delta)}{n} < \frac{\ln(M)}{\ln(1/\delta)}.$$

The result follows.

4. The points inside the box which are distance at least 1 from all of the sides form a rectangular box with sides 1,2,3, which has volume 6. The volume of the original box is 60. The points outside the box which are distance at most 1 from one of the sides have volume

$$3\times4+3\times4+3\times5+3\times5+4\times5+4\times5 = 94$$

plus the points at the corners, which form eight $\frac{1}{8}$ th spheres of radius 1, plus the points which form 12 $\frac{1}{4}$ th cylinders whose heights are 3,4,5. It follows that the volume required is

$$60-6+94+4\pi/3+12\pi = 148+40\pi/3.$$

5. By differentiating $f(f(x)) = x$, we obtain $f'(f(x))f'(x) = 1$. Since f is continuous, $f'(x)$ can never cross zero. This means that either $f'(x) > 0$ for all x of $f'(x) < 0$ for all x. If $f'(x) > 0$ for all x, then $x > y$ implies $f(x) > f(y)$, and we get a contradiction by considering $f(f(a)) = a$. We deduce that f is monotonically decreasing, and since f is bounded below by 0, we see that $\lim_{x\to\infty} f(x)$ exists and is some nonnegative number, which we shall call L. If $L > 0$, then we obtain a contradiction by considering $f(f(L/2)) = L/2$. The result follows. **Remark** The condition $f(a) \neq a$ is required, otherwise $f(x) = x$ would be a solution.

6. (i) Obviously $n > 4$. Next, $n \neq 5$ because 4 divides $3+5$. Also $n \neq 6$ because 3 divides 6 and $n \neq 7$ because 7 divides $3+4$. Finally $n \neq 8$ since 4 divides 8, and $n \neq 9$ since 3 divides 9. On the other hand $n = 10$ because 3 does not divide 4, 10 and 14. Furthermore 4 does not divide 3, 10 and 13, and 10 does not divide 3, 4 and 7.

(ii) Suppose $\{3, 4, 10, m\}$ is contained in a set which has property **ND**. Then 3 should not divide m, so m is not of the form $3k$. Also 3 should not divide $10 + m$, so m is not of the form $3k + 2$. Furthermore 33 should not divide $m + 4 + 10$, so m is not of the form $3k + 1$. Here k denotes some integer. If s has property **ND** and contains 3,4 10 and m, then m cannot be of the form $3k$, $3k + 1$, $3k + 2$. This is impossible and the statement is proven.