20th VTRMC, 1998, Solutions

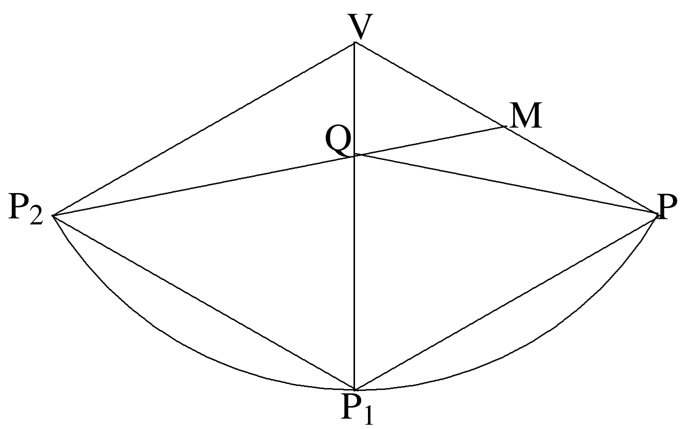
1. Set $r = x^2 + y^2$. Then $f(x,y) = \ln(1-r) - 1/(2r - (x+y)^2)$, so for given r, we see that f is maximized when $x + y = 0$. Therefore we need to maximize $\ln(1-r) - 1/(2r)$ where $0 < r < 1$. The derivative of this function is

$$\frac{1}{r-1} + \frac{1}{2r^2} = \frac{2r^2 + r - 1}{2(r-1)r^2}$$

which is positive when $r < 1/2$, 0 when r=1/2, and negative when $r > 1/2$. It follows that the maximum value of this function occurs when $r = 1/2$ and we deduce that $M = -1 - \ln 2$.

2. We cut the cone along PV and then open it out flat, so in the picture below P and P_1 are the same point. We want to find Q on VP_1 so that the length of MQP is minimal. To do this we reflect in VP_1 so P_2 is the image of P under this reflection, and then MQP_2 will be a straight line and the problem is to find the length of MP_2.

   Since the radius of the base of the cone is 1, we see that the length from P to P_1 along the circular arc is 2π, hence the angle $\angle PVP_1$ is $\pi/3$ because $VP = 6$. We deduce that $\angle PVP_2 = 2\pi/3$, and since $VM = 3$ and $VP_2 = VP = 6$, we conclude that $MP_2 = \sqrt{3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cdot \cos(2\pi/3)} = 3\sqrt{7}$.

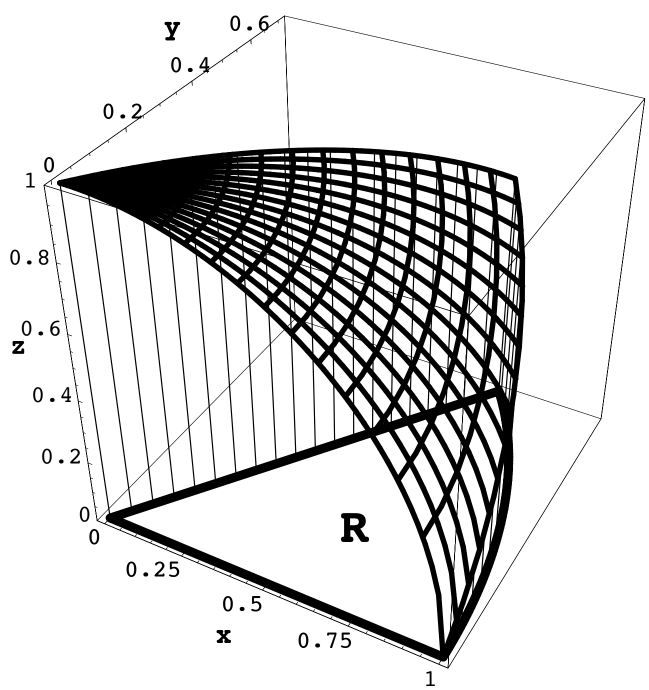


3. We calculate the volume of the region which is in the first octant and above $\{(x,y,0) \mid x \geq y\}$; this is 1/16 of the required volume. The volume is above R, where R is the region in the xy-plane and bounded by $y = 0$, $y = x$ and

$y=\sqrt{1-x^2}$, and below $z=\sqrt{1-x^2}$. This volume is

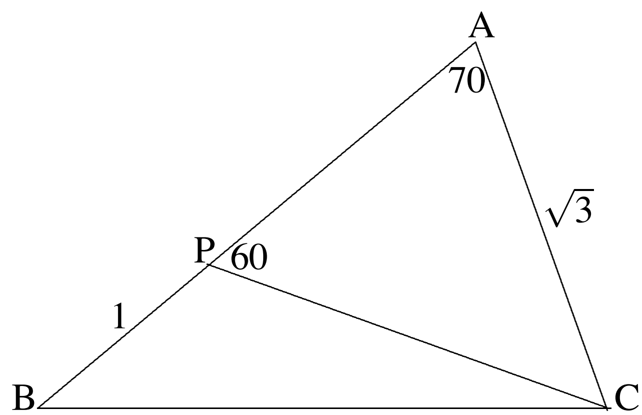
$$\begin{aligned}
&\int_0^{1/\sqrt{2}}\int_0^x\int_0^{\sqrt{1-x^2}} dzdydx+\int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx\\
&=\int_0^{1/\sqrt{2}}\int_0^x \sqrt{1-x^2}\,dydx+\int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dydx\\
&=\int_0^{1/\sqrt{2}} x\sqrt{1-x^2}\,dx+\int_{1/\sqrt{2}}^1 (1-x^2)\,dx\\
&=[-(1-x^2)^{3/2}/3]_0^{1/\sqrt{2}}+[x-x^3/3]_{1/\sqrt{2}}^1\\
&=\frac{1}{3}-\frac{1}{6\sqrt{2}}+\frac{2}{3}+\frac{1}{6\sqrt{2}}-\frac{1}{\sqrt{2}}\\
&=1-1/\sqrt{2}.
\end{aligned}$$

Therefore the required volume is $16-8\sqrt{2}$.



4. We shall prove that $AB=BC$. Using the cosine rule applied to the triangle ABC, we see that $BC^2=AB^2+AC^2-2(AB)(AC)\cos 70$. Therefore we need to prove $AC=2AB\cos 70$. By the sine rule applied to the triangle APC, we

find that $AP = 2\sin 50$, so we need to prove $\sqrt{3} = 2(1 + 2\sin 50)\cos 70$. However $\sin(50+70) + \sin(50-70) = 2\sin 50 \cos 70$, $\sin 120 = \sqrt{3}/2$ and $\sin(50-70) = -\cos 70$. The result follows.



5. Since $\sum 1/a_n$ is a convergent series of positive terms, we see that given $M > 0$, there are only finitely many positive integers n such that $a_n < M$. Also rearranging a series with positive terms does not affect its convergence, hence we may assume that $\{a_n\}$ is a monotonic increasing sequence. Then $b_{2n+1} \geq b_{2n} \geq a_n/2$, so the terms of the sequence $\{1/b_n\}$ are at most the corresponding terms of the sequence

$$\frac{2}{a_1}, \frac{2}{a_1}, \frac{2}{a_2}, \frac{2}{a_2}, \frac{2}{a_3}, \frac{2}{a_3}, \dots$$

Since $\sum 1/a_n$ is convergent, so is the sum of the above sequence and the result now follows from the comparison test for positive term series.

6. We shall assume the theory of writing permutations as a product of disjoint cycles, though this is not necessary. Rule 1 corresponds to the permutation (1 2 3 4 5 6 7 8 9 10) and Rule 2 corresponds to the permutation (2 6)(3 4)(5 9)(7 8). Since

$$(2\ 6)(3\ 4)(5\ 9)(7\ 8)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10) = (1\ 6\ 8\ 5\ 2\ 4\ 9\ 10)$$

(where we have written mappings on the left) has order 8, we see the position of the cats repeats once every 16 jumps. Now 10 p.m. occurs after 900 jumps, hence the cats are in the same position then as after 4 jumps and we conclude that the white cat is on post 8 at 10 p.m.