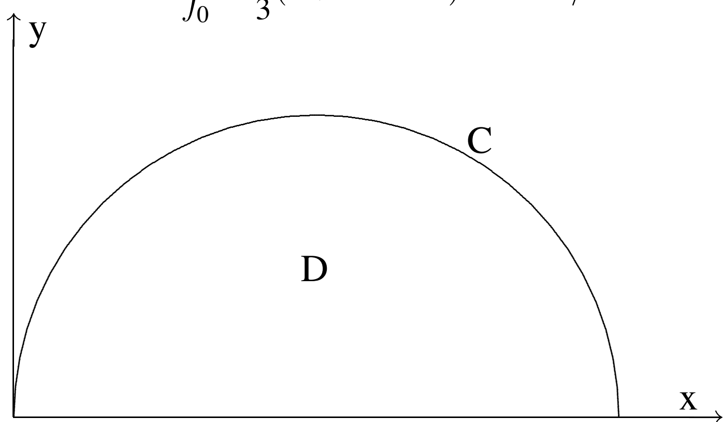
19th VTRMC, 1997, Solutions

1. We change to polar coordinates. Thus $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr d\theta$. The circle $(x-1)^2+y^2 = 1$ becomes $r^2 - 2r\cos\theta = 0$, which simplifies to $r = 2\cos\theta$. Also as one moves from $(2,0)$ to $(0,0)$ on the semicircle C (see diagram below), θ moves from 0 to $\pi/2$. Therefore

$$\iint_D \frac{x^3}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r^3\cos^3\theta}{r^2} r\,dr d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos^3\theta\,dr d\theta$$
$$= \int_0^{\pi/2} \frac{8}{3}\cos^6\theta\,d\theta = \int_0^{\pi/2} \frac{1}{3}(1+\cos 2\theta)^3\,d\theta$$
$$= \int_0^{\pi/2} \frac{1}{3}(1+3\cos^2 2\theta)\,d\theta = 5\pi/12.$$



2. Since $r_1r_2 = 2$, the roots r_1, r_2 will satisfy a quadratic equation of the form $x^2+px+2 = 0$, where $p \in \mathbb{C}$. Therefore we may factor

$$x^4 - x^3 + ax^2 - 8x - 8 = (x^2+px+2)(x^2+qx-4)$$

where $q \in \mathbb{C}$. Equating the coefficients of x^3 and x, we obtain $p+q = -1$ and $2q-4p = -8$. Therefore $p = 1$ and $q = -2$. We conclude that $a = -4$ and r_1, r_2 are the roots of x^2+x+2, so r_1 and r_2 are $(-1 \pm i\sqrt{7})/2$.

3. The number of different combinations of possible flavors is the same as the coefficient of x^{100} in

$$(1+x+x^2+\dots)^4$$

This is the coefficient of x^{100} in $(1-x)^{-4}$, that is $103!/(3!100!) = 176851$.

4. We can represent the possible itineraries with a matrix. Thus we let

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

and let a_{ij} indicate the (i,j)th entry of A. Then for a one day period, a_{11} is the number of itineraries from New York to New York, a_{12} is the number of itineraries from New York to Los Angeles, a_{21} is the number of itineraries from Los Angeles to New York, and a_{22} is the number of itineraries from Los Angeles to Los Angeles. The number of itineraries for an n day period will be given by A^n; in particular the $(1,1)$ entry of A^{100} will be the number of itineraries starting and finishing at New York for a 100 day period.

To calculate A^{100}, we diagonalize it. Then the eigenvalues of A are $1 \pm \sqrt{3}$ and the corresponding eigenvectors (vectors $\mathbf{u}$ satisfying $A\mathbf{u} = \lambda\mathbf{u}$ where $\lambda = 1 \pm \sqrt{3}$) are $(\pm\sqrt{3}, 1)$. Therefore if $P = \begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix}$, then

$$P^{-1}AP = \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix}$$

Thus

$$A^{100} = P \begin{pmatrix} (1+\sqrt{3})^{100} & 0 \\ 0 & (1-\sqrt{3})^{100} \end{pmatrix} P^{-1}.$$

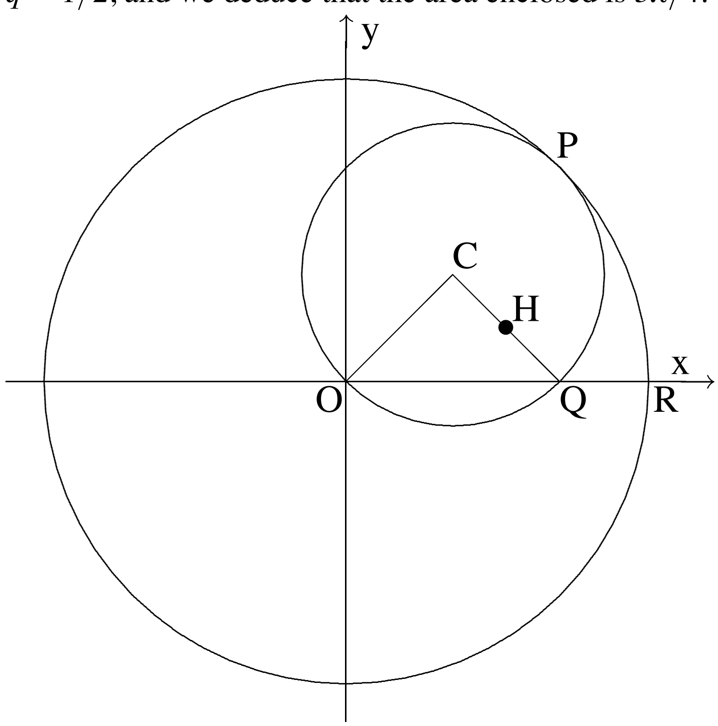
We conclude that the $(1,1)$ entry of A^{100} is $((1+\sqrt{3})^{100} + (1-\sqrt{3})^{100})/2$, which is the number of itineraries required.

5. For each city x in S, let $G_x \subset S$ denote all the cities which you can travel from x (this includes x). Clearly G_x is well served and $|G_x| \geq 3$ (where $|G_x|$ is the number of cities in G_x). Choose x so that $|G_x|$ is minimal. We need to show that if $y, z \in G_x$, then one can travel from y to z stopping only at cities in G_x; clearly we need only prove this in the case $z = x$. So suppose by way of contradiction $y \in G_x$ and we cannot travel from y to x stopping only at cities in G_x. Since $G_y \subseteq G_x$ and $x \notin G_y$, we have $|G_y| < |G_x|$, contradicting the minimality of $|G_x|$ and the result follows.

6. Let O denote the center of the circle with radius 2 cm., let C denote the center of the disk with radius 1 cm., and let H denote the hole in the center

of the disk. Choose axes so that the origin is at O, and then let the initial position have C and H on the positive x-axis with H furthest from O. The diagram below is in general position (i.e. after the disk has been moved round the inside of the circle). Let P be the point of contact of the circle and the disk, (so OCP will be a straight line), let Q be where CH meets the circumference of the disk (on the x-axis, though we need to prove that), and let R be where the circle meets the positive x-axis. Since the arc lengths PQ and PR are equal and the circle has twice the radius of that of the disk, we see that $\angle PCQ = 2\angle POR$ and it follows that Q does indeed lie on the x-axis.

Let (a,b) be the coordinates of C. Then $a^2+b^2=1$ because the disk has radius 1, and the coordinates of H are $(3a/2, b/2)$. It follows that the curve H traces out is the ellipse $4x^2+36y^2=9$. We now use the formula that the area of an ellipse with axes of length $2p$ and $2q$ is πpq. Here $p=3/2$, $q=1/2$, and we deduce that the area enclosed is $3\pi/4$.



7. Let $x = \{x_0, x_1, \ldots, x_n\} \in J$. Then

$$\begin{aligned}
Tx &= LA(\{x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots\}) \\
&= L(\{1 + x_0, 1 + x_0 + x_1, 1 + x_0 + x_1 + x_2, \ldots\}) \\
&= \{1, 1 + x_0, 1 + x_0 + x_1, 1 + x_0 + x_1 + x_2, \ldots\}.
\end{aligned}$$

Therefore $T^2 y = T(\{1,2,3,\ldots\}) = \{1, 1+1, 1+1+2, 1+1+2+3, \ldots\}$. We deduce that $T^2 y = \{1,2,4,7,11,16,22,29,\ldots\}$ and in general $(T^2 y)_n = n(n+1)/2 + 1$.

Suppose $z = \lim_{i\to\infty} T^i y$ exists. Then $Tz = z$, so $1 = z_0$, $1 + z_0 = z_1$, $1 + z_0 + z_1 = z_2$, $1 + z_0 + z_1 + z_2 = z_3$, etc. We now see that $z_n = 2^n$. To verify this, we use induction on n, the case $n = 0$ already having been established. Assume true for n; then

$$z_{n+1} = 1 + z_0 + z_1 + \cdots + z_n = 1 + 1 + 2 + \cdots + 2^n = 2^{n+1},$$

so the induction step is complete and the result is proven.