18th VTRMC, 1996, Solutions

1. Let $I = \int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^4+2x^2y^2+y^4)}\,dxdy$. We change to polar coordinates to obtain
$$I = \int_0^{\pi/2}\int_{\sin\theta}^1 r\cos\theta\, e^{r^4}\, rdrd\theta = \int_0^{\pi/2}\int_{\sin\theta}^1 r^2 e^{r^4}\cos\theta\, drd\theta.$$
Now we reverse the order of integration; also we shall write $t = \theta$. This yields
$$\begin{aligned} I &= \int_0^1 \int_0^{\sin^{-1} r} r^2 e^{r^4}\cos t\, dtdr = \int_0^1 [r^2 e^{r^4}\sin t]_0^{\sin^{-1} r}\, dr \\ &= \int_0^1 r^3 e^{r^4}\, dr = [e^{r^4}/4]_0^1 = (e-1)/4. \end{aligned}$$

2. Write $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$, where m_1, n_1, m_2, n_2 are positive integers and $\gcd(m_1, n_1) = 1 = \gcd(m_2, n_2)$. Set $Q = ((m_1+m_2)/(n_1+n_2), 1/(n_1+n_2))$. We note that Q is on the line joining $(r_1, 0)$ with $P(r_2)$, that is the line joining $(m_1/n_1, 0)$ with $(m_2/n_2, 1/n_2)$. This is because
$$(m_1+m_2)/(n_1+n_2) = m_1/n_1 + (m_2/n_2 - m_1/n_1)(n_2/(n_1+n_2)).$$
Similarly Q is on the line joining $P(r_1)$ with $(r_2, 0)$. It follows that $(m_1+m_2)/(n_1+n_2), 1/(n_1+n_2)$ is the intersection of the line joining $(r_1, 0)$ to $P(r_2)$ and the line joining $P(r_1)$ and $(r_2, 0)$. Set
$$P = P((r_1 f(r_1) + r_2 f(r_2))/(f(r_1)+f(r_2))).$$
Since
$$P = ((m_1+m_2)/(n_1+n_2), /f((m_1+m_2)/(n_1+n_2))),$$
we find that P is the point of intersection of the two given lines if and only if $f((m_1+m_2)/(n_1+n_2)) = n_1+n_2$. We conclude that the necessary and sufficient condition required is that $\gcd(m_1+m_2, n_1+n_2) = 1$.

3. Taking logs, we get $dy/dx = y\ln y$, hence $dx/dy = 1/(y\ln y)$. Integrating both sides, we obtain $x = \ln(\ln y) + C$ where C is an arbitrary constant. Plugging in the initial condition $y = e$ when $x = 1$, we find that $C = 1$. Thus $\ln(\ln y) = x - 1$ and we conclude that $y = e^{(e^{x-1})}$.

4. Set $g(x) = x^2 f(x)$. Then the given limit says $\lim_{x\to\infty} g''(x) = 1$. Therefore $\lim_{x\to\infty} g'(x) = \lim_{x\to\infty} g(x) = \infty$. Thus by l'Hôpital's rule,

$$\lim_{x\to\infty} g(x)/x^2 = \lim_{x\to\infty} g'(x)/(2x) = \lim_{x\to\infty} g(x)/2 = 1/2.$$

We deduce that $\lim_{x\to\infty} f(x) = 1/2$ and $\lim_{x\to\infty}(xf'(x)/2 + f(x)) = 1/2$, and the result follows.

5. Set

$$f(x) = a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6,$$
$$g(x) = a_1x + b_1x^2/2 + a_2x^3 + b_2x^4/4 + a_3x^5 + b_3x^6/6 + a_4x^7.$$

Then $g(1) = g(-1)$ because $a_1 + a_2 + a_3 + a_4 = 0$, hence there exists $t \in (-1,1)$ such that $g'(t) = 0$. But $g'(x) = f(x)$ and the result follows.

6. We choose the n line segments so that the sum of their lengths is as small as possible. We claim that no two line segments intersect. Indeed suppose A, B are red balls and C, D are green balls, and AC intersects BD at the point P. Since the length of one side of a triangle is less than the sum of the lengths of the two other sides, we have $AD < AP + PD$ and $BC < BP + PC$, consequently

$$AD + BC < AP + PC + BP + PD = AC + BD,$$

and we have obtained a setup with the sum of the lengths of the line segments strictly smaller. This proves that the line segments can be chosen so that no two intersect.

7. We have $f_{n,j+1}(x) - f_{n,j}(x) = \sqrt{x}/n$, hence

$$f_{n,j}(x) = f_{0,j}(x) + j\sqrt{x}/n = x + (j+1)\sqrt{x}/n.$$

Thus in particular $f_{n,n}(x) = x + (n+1)\sqrt{x}/n$ and we see that $\lim_{n\to\infty} f_{n,n}(x) = x + \sqrt{x}$.