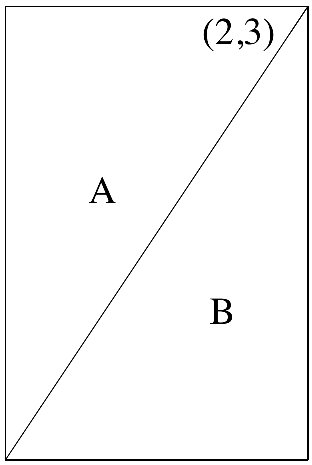
17th VTRMC, 1995, Solutions

1. Let $A = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3,\ 3x \le 2y\}$ and $B = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3,\ 3x \le 2y\}$. Let $I = \int_0^3 \int_0^2 1/(1+\max(3x,2y))^2\, dxdy$. Then $\max(3x,2y) = 2y$ for $(x,y) \in A$ and $\max(3x,2y) = 3x$ for $x \in B$.



Therefore

$$\begin{aligned} I &= \iint_A 1/(1+2y)^2\, dA + \iint_B 1/(1+3x)^2\, dA \\ &= \int_0^3 \int_0^{2y/3} 1/(1+2y)^2\, dxdy + \int_0^2 \int_0^{3x/2} 1/(1+3x)^2\, dydx \\ &= \int_0^3 2y/(3(1+2y)^2)\, dy + \int_0^2 3x/(2(1+3x)^2)\, dx \\ &= \int_0^3 1/(3(1+2y)) - 1/(3(1+2y)^2)\, dy \\ &\qquad + \int_0^2 1/(2(1+3x)) - 1/(2(1+3x)^2)\, dx \\ &= [(\ln(1+2y))/6 + 1/(6(1+2y))]_0^3 + [(\ln(1+3x))/6 + 1/(6(1+3x))]_0^2 \\ &= (\ln 7)/6 + 1/42 - 1/6 + (\ln 7)/6 + 1/42 - 1/6 - (7\ln 7 - 6)/21. \end{aligned}$$

2. Let $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$. We want to calculate powers of A, and to do this it is useful to find the Jordan canonical form of A. The characteristic polynomial of A is $\det(xI - A) = (x-4)(x+1)+6 = x^2 - 3x + 2$ which has roots 1,2. Set $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. An eigenvector corresponding to 1 is $\mathbf{u}$ and an eigenvector

corresponding to 2 is $\begin{pmatrix}3\\2\end{pmatrix}$. Set $P = \begin{pmatrix}1 & 3\\1 & 2\end{pmatrix}$ and $D = \operatorname{diag}(1,2)$ (diagonal matrix with 1,2 on the main diagonal). Then $P^{-1} = \begin{pmatrix}-2 & 3\\1 & -1\end{pmatrix}$ and $P^{-1}AP = D$. Let $\mathbf{v} = \begin{pmatrix}1\\0\end{pmatrix}$ and let ^{T} denote transpose. Since $A = PDP^{-1}$, we find that

$$\begin{aligned}(\theta^{100}\mathbf{v})^T &= A^{100}\mathbf{v} + (A^{99} + A^{98} + \cdots + A + A^0)\mathbf{u}\\ &= PD^{100}P^{-1}\mathbf{v} + P(D^{99} + D^{98} + \cdots + D + D^0)P^{-1}\mathbf{u}\\ &= P\begin{pmatrix}1 & 0\\0 & 2^{100}\end{pmatrix}P^{-1}\mathbf{v} + P\begin{pmatrix}100 & 0\\0 & 2^{100}-1\end{pmatrix}P^{-1}\mathbf{u}\\ &= \begin{pmatrix}98 + 3\cdot 2^{100}\\98 + 2\cdot 2^{100}\end{pmatrix}.\end{aligned}$$

Thus $\theta^{100}(1,0) = (98 + 3\cdot 2^{100}, 98 + 2\cdot 2^{100})$.

3. Let $g(x)$ denote the power series in x

$$1 - (x + x^2 + \cdots + x^n) + (x + x^2 + \cdots + x^n)^2 - \cdots + (-1)^n(x + x^2 + \cdots + x^n)^n + \cdots.$$

Then for $2 \le r \le n$, the coefficient of x^r in $f(x)$ is the same as the coefficient of x^r in $g(x)$. Since $x + x^2 + \cdots + x^n = x(1 - x^n)/(1 - x)$, we see that $g(x)$ is a geometric series with ratio between successive terms $-x(1 - x^n)/(1 - x)$, hence its sum is

$$\frac{1}{1 + x(1 - x^n)/(1 - x)} = \frac{1 - x}{1 - x^{n+1}} = (1 - x)(1 + x^{n+1} + x^{2n+2} + \cdots).$$

clearly the coefficient of x^r in the above is 0 for $2 \le r \le n$, which proves the result.

4. Write $[\tau n] = p$. Then p is the unique integer satisfying $p < \tau n < p + 1$ because $p \ne \tau n$ (otherwise $\tau = p/n$, a rational number), that is $p/\tau < n < p/\tau + 1$. Since $1/\tau = \tau - 1$, we see that $p\tau - p < n < p\tau - p + 1$ and we deduce that $n + p - 1 < p\tau < n + p$. Therefore $[p\tau] = n + p - 1$ and hence $[\tau[\tau n] + 1] = n + p$. But $\tau^2 n = \tau n + n$, consequently $[\tau^2 n] = p + n$ and the result follows.

5. Suppose $x \in \mathbb{R}$ and $\theta(x) \leq -1$. Fix $y \in \mathbb{R}$ with $y < x$. Then if n is a positive integer and $x > p_1 > \cdots > p_n > y$, we have for $1 \leq i \leq n$

$$\begin{aligned} \theta(x) &\geq \theta(x)^3 > \theta(p_1), \\ \theta(p_i) &> \theta(p_i)^3 > \theta(p_{i+1}), \\ \theta(p_n) &> \theta(p_n)^3 > \theta(y), \end{aligned}$$

and we deduce that $\theta(x)\theta(p_1)^{2n-2} > \theta(y)$, for all n. this is not possible, so $\theta(x) > -1$ for all $x \in \mathbb{R}$. The same argument works if $0 \leq \theta(y) < \theta(x) \leq 1$.

6. We will concentrate on the bottom left hand corner of the square and determine the area A of that portion of the square that can be painted by the brush, and then multiply that by 4. We make the bottom of the square the x-axis and the left hand side of the square the y-axis. The equation of a line of length 4 from $(a,0)$ to the y-axis is $x/a + y/\sqrt{16-a^2} = 1$, that is $y = (1-x/a)\sqrt{16-a^2}$. For fixed x, we want to know the maximum value y can take by varying a. To do this, we differentiate y with respect to a and then set the resulting expression to 0. Thus we need to solve

$$(x/a^2)\sqrt{16-a^2} - a(1-x/a)/\sqrt{16-a^2} = 0.$$

On multiplying by $\sqrt{16-a^2}$ and simplifying, we obtain $16x = a^3$ and hence $dx/da = 3a^2/16$. Therefore

$$\begin{aligned} A &= \int_{x=0}^{x=4} (1-x/a)\sqrt{16-a^2}\,dx = \int_{a=0}^{a=4} (1-x/a)\sqrt{16-a^2}\,\frac{dx}{da}da \\ &= \int_{a=0}^{a=4} 3a^2(1-a^2/16)\sqrt{16-a^2}/16\,da = \int_0^4 3a^2(16-a^2)^{3/2}/256\,da. \end{aligned}$$

This ia a standard integral which can be evaluated by a trigonometric substitution. Specifically we set $a = 4\sin t$, so $da/dt = 4\cos t$ and we find that

$$\begin{aligned} A &= \int_0^{\pi/2} 48\cos^4 t \sin^2 t\,dt = \int_0^{\pi/2} 6\sin^2 2t(1+\cos 2t)\,dt \\ &= \int_0^{\pi/2} 3(1-\cos 4t)\,dt = 3\pi/2. \end{aligned}$$

We conclude that the total area that can be painted by the brush is $6\pi\,\mathrm{in}^2$.

7. Note that if p is a prime, then $f(p) = p$. Thus $f(100) = f(2^2 \cdot 5^2) = 4 + 10 = 14$, $f(2 \cdot 7) = 2 + 7 = 9$, $f(3^2) = 3 \cdot 2 = 6$. Therefore $g(100) = 6$. Next $f(10^{10}) = f(2^{10} \cdot 5^{10}) = 20 + 50 = 70$, $f(2 \cdot 5 \cdot 7) = 14$, $f(2 \cdot 7) = 2 + 7 = 9$, $f(3^2) = 3 \cdot 2 = 6$. Therefore $g(10^{10}) = 6$.

   Since $f(p) = p$ if p is prime, we see that $g(p) = p$ also and thus primes cannot have property H. Note that if r, s are coprime, then $g(rs) \leq f(r)s$. Suppose n has property H and let p be a prime such that p^2 divides n, so $n = p^k r$ where $k \geq 2$ and r is prime to p. It is easy to check that if $p^k > 9$, then $p^k > 2pk$, that is $f(p^k) < p^k/2$, thus $f(n) < n/2$ and we see that n cannot have property H. Also if p, q are distinct odd primes and $pq > 15$, then $f(pq) < pq/2$ and so if $n = pqr$ with r prime to pq, then we see again that n cannot have property H.

   The only cases to be considered now are $n = 9, 15, 45$. By direct calculation, 9 has property H, but 15 and 45 do not. So the only positive odd integer larger than 1 that has property H is 9.