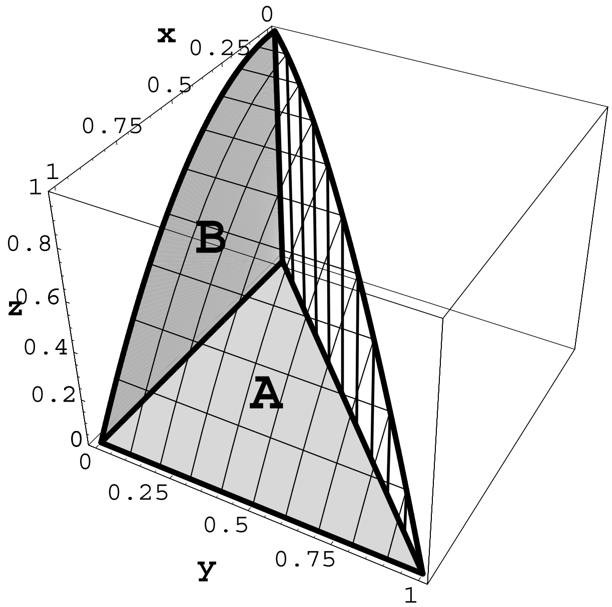
16th VTRMC, 1994, Solutions

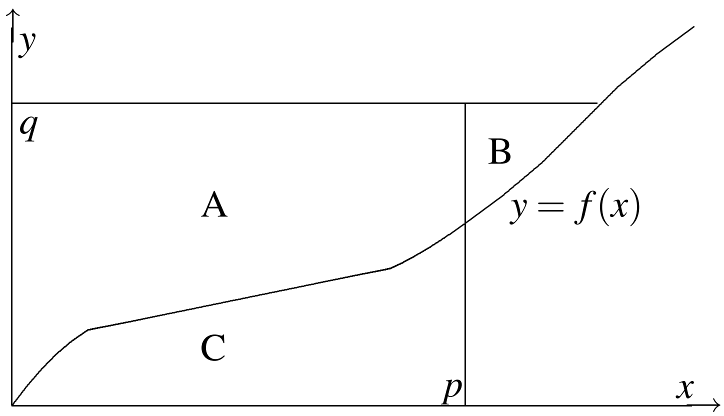
1. Let $I = \int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2}\, dz dy dx$. We change the order of integration, so we write $I = \iiint_V e^{(1-z)^2}\, dV$, where V is the region of integration.



It can be described as the cylinder with axis parallel to the z-axis and cross-section A, bounded below by $z = 0$ and bounded above by $z = 1 - x^2$. This region can also be described as the cylinder with axis parallel to the y-axis and cross-section B, bounded on the left by $y = 0$ and on the right by $y = x$. Therefore

$$
\begin{aligned}
I &= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^x e^{(1-z)^2}\, dy dx dz \\
&= \int_0^1 \int_0^{\sqrt{1-z}} x e^{(1-z)^2}\, dx dz = \int_0^1 (1-z) e^{(1-z)^2}/2\, dz \\
&= [-e^{(1-z)^2}/4]_0^1 = (e-1)/4.
\end{aligned}
$$

2. We need to prove that $pq \le \int_0^p f(t)\, dt + \int_0^q g(t)\, dt$. Either $q \le f(p)$ or $q \ge f(p)$ and without loss of generality we may assume that $q \ge f(p)$ (if $q \le f(p)$, then we interchange x and y; alternatively just follow a similar argument to what is given below). Then we have the following diagram.



We now interpret the quantities in terms of areas: pq is the area of $A \cup C$, $\int_0^p f(t)\,dt$ is the area of C, and $\int_0^q g(t)\,dt$ is the area of $A \cup B$. The result follows.

3. Differentiating both sides with respect to x, we obtain $2ff' = f^2 - f^4 + (f')^2$. Thus $f^4 = (f - f')^2$, hence $f - f' = \pm f^2$ and we deduce that $dx/df = \frac{1}{f \pm f^2}$. We have two cases to consider; first we consider the + sign, that is $dx/df = 1/f - 1/(f+1)$ and we obtain $x = \ln|f| - \ln|f+1| + C$, where C is an arbitrary constant. Now we have the initial condition $f(0) = \pm 10$. If $f(0) = 10$, we find that $C = \ln(11/10)$ and consequently $x = \ln(11/10) - \ln|(f+1)|/f|$. Solving this for x, we see that $f(x) = 10/(11e^{-x} - 10)$. On the other hand if $f(0) = -10$, then $C = \ln(9/10)$, consequently $x = \ln(9/10) - \ln|(f+1)/f|$. Solving this for x, we conclude that $f(x) = 10/(9e^{-x} - 10)$.

   Now we consider the – sign, that is $dx/df = 1/f - 1/(f-1)$ and we obtain $x = \ln|f| - \ln|f-1| + D$, where D is an arbitrary constant. If the initial condition $f(0) = 10$, we find that $D = \ln(9/10)$ and consequently $x = \ln|f/(f-1)| + \ln(9/10)$. Solving this for x, we deduce that $f(x) = 10/(10 - 9e^{-x})$. On the other hand if the initial condition is $f(0) = -10$, then $D = \ln(11/10)$ and hence $x = \ln|f/(f-1)| + \ln(11/10)$. Solving for x, we conclude that $f(x) = 10/(10 - 11e^{-x})$.

   Summing up, we have

$$f(x) = \frac{\pm 10}{10 - 9e^{-x}} \quad \text{or} \quad \frac{\pm 10}{10 - 11e^{-x}}.$$

4. Set $f(x) = ax^4 + bx^3 + x^2 + bx + a = 0$. We will show that the maximum value of $a + b$ is $-1/2$; certainly $-1/2$ can be obtained, e.g. with $a = 1$

corresponding first order differential equation $y' = 4y + 4t$. The solution to $a_{n+1} = 4a_n$ is $a_n = C4^n$ for some constant n. Then we look for a solution to $a_{n+1} = 4a_n + 4n$ in the form $a_n = An + B$, where A, B are constants to be determined. Plugging this into the recurrence relation, we obtain $A(n+1) + B = 4An + 4B + 4n$, and then equating the coefficients of n and the constant term, we find that $A = -4/3, B = -4/9$. Therefore $a_n = C4^n - 4n/3 - 4/9$, and then plugging in $a_1 = 1$, we see that $C = 25/36$ and we conclude that $a_n = 25 \cdot 4^n/36 - 4n/3 - 4/9$. We now need to calculate $\sum_{n=1}^{N} a_n$. This is

$$25(4^N - 1)/27 - 2N^2/3 - 10N/9.$$

8. We have

$$x_{n+3} = \frac{19x_{n+2}}{94_{n+1}} = \frac{19^2}{94^2 x_n}$$

and we deduce that $x_{n+6} = x_n$ for all nonnegative integers n. It follows that $\sum_{n=0}^{\infty} x_{6n}/2^n = \sum_{n=0}^{\infty} 10/2^n = 20$.