13th VTRMC, 1991, Solutions

1. Let P denote the center of the circle. Then $\angle ACP = \angle ABP = \pi/2$ and $\angle BAP = \alpha/2$. Therefore $BP = a\tan(\alpha/2)$ and we see that $ABPC$ has area $a^2\tan(\alpha/2)$. Since $\angle BPC = \pi - \alpha$, we find that the area of the sector BPC is $(\pi/2 - \alpha/2)a^2\tan^2(\alpha/2)$. Therefore the area of the curvilinear triangle is
$$a^2\left(1+\frac{\alpha}{2}-\frac{\pi}{2}\right)\tan^2\frac{\alpha}{2}.$$

2. If we differentiate both sides with respect to x, we obtain $3f(x)^2f'(x) = f(x)^2$. Therefore $f(x) = 0$ or $f'(x) = 1/3$. In the latter case, $f(x) = x/3 + C$ where C is a constant. However $f(0)^3 = 0$ and we see that $C = 0$. We conclude that $f(x) = 0$ and $f(x) = x/3$ are the functions required.

3. We are given that α satisfies $(1+x)x^{n+1} = 1$, and we want to show that α satisfies $(1+x)x^{n+2} = x$. This is clear, by multiplying the first equation by x.

4. Set $f(x) = x^n/(x+1)^{n+1}$, the left hand side of the inequality. Then
$$f'(x) = \frac{x^{n-1}}{(x+1)^{n+2}}(n-x).$$
This shows, for $x > 0$, that $f(x)$ has its maximum value when $x = n$ and we deduce that $f(x) \le n^n/(n+1)^{n+1}$ for all $x > 0$.

5. Clearly there exists c such that $f(x) - c$ has a root of multiplicity 1, e.g. $x = c = 0$. Suppose $f(x) - c$ has a multiple root r. Then r will also be a root of $(f(x)-c)' = 5x^4 - 15x^2 + 4$. Also if r is a triple root of $f(x) - c$, then it will be a double root of this polynomial. But the roots of $5x^4 - 15x^2 + 4$ are $\pm((15 \pm \sqrt{145})/10)^{1/2}$, and we conclude that $f(x) - c$ can have double roots, but neither triple nor quadruple roots.

6. Expand $(1-1)^n$ by the binomial theorem and divide by $n!$. We obtain for $n > 0$
$$\frac{1}{0!n!} - \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!} - \cdots + \frac{(-1)^n}{n!0!} = 0.$$
Clearly the result is true for $n = 0$. We can now proceed by induction; we assume that the result is true for positive integers $< n$ and plug into the

above formula. We find that

$$\frac{a_0}{n!}+\frac{a_1}{(n-1)!}+\frac{a_2}{(n-2)!}+\cdots+\frac{a_{n-1}}{1!}+\frac{(-1)^n}{n!0!}=0$$

and the result follows.

7. Suppose $2/3 < a_n, b_n < 7/6$. Then $2/3 < a_{n+2}, b_{n+2} < 7/6$. Now if $c = 1.26$, then $2/3 < a_3, b_3 < 1$, so if $x_n = a_{2n+1}$ or b_{2n+1}, then $x_{n+1} = x_n/4 + 1/2$ for all $n \geq 1$. This has the general solution of the form $x_n = C(1/4)^n + 2/3$. We deduce that as $n \to \infty$, a_{2n+1}, b_{2n+1} decrease monotonically with limit 2/3, and a_{2n}, b_{2n} decrease monotonically with limit 4/3.

   On the other hand suppose $a_n > 3/2$ and $b_n < 1/2$. Then $a_{n+1} > 3/2$ and $b_{n+1} < 1/2$. Now if $c = 1.24$, then $a_3 > 3/2$ and $b_3 < 1/2$. We deduce that $a_{n+1} = a_n/2 + 1$ and $b_{n+1} = b_n/2$. This has general solution $a_n = C(1/2)^n + 2$, $b_n = D(1/2)^n$. We conclude that as $n \to \infty$, a_n increases monotonically to 2 and b_n decreases monotonically to 0.

8. Let A be a base campsite and let h be a hike starting and finishing at A which covers each segment exactly once. Let B be the first campsite which h visits twice (i.e. B is the earliest campsite that h reaches a second time). This could be A after all segments have been covered, and then we are finished (just choose $\mathscr{C} = \{h\}$). Otherwise let h_1 be the hike which is the part of h which starts with the first visit to B and ends with the second visit to B (so B is the base campsite for h_1). Let h' be the hike obtained from h by omitting h_1 (so h' doesn't visit all segments). Now do the same with h'; let C be the first campsite on h' (starting from A) that is visited twice and let h_2 be the hike which is the part of h' that starts with the first visit to C and ends with the second visit to C. Then $\mathscr{C}$ can be chosen to be the collection of hikes $\{h_1, h_2, \dots\}$ to do what is required.