12th VTRMC, 1990, Solutions

1. Let a be the initial thickness of the grass, let b the rate of growth of the grass, and let c be the rate at which the cows eat the grass (in the appropriate units). Let n denote the number of cows that will eat the third field bare in 18 weeks. Then we have

$$\begin{aligned} 10(a+4b)/3 &= 12*4c \\ 10(a+9b) &= 21*9c \\ 24(a+18b) &= n18c \end{aligned}$$

If we multiply the first equation by $-27/5$ and the second equation by 14/5, we obtain $10(a+18b) = 270c$, so $(a+18b)/c = 27$. We conclude that $n = 36$, so the answer is 36 happy cows.

2. The exact number N of minutes to complete the puzzle is $\sum_{x=0}^{999} 3(1000-x)/(1000+x)$. Since $3(1000-x)/(1000+x)$ is a non-negative monotonic decreasing function for $0 \leq x \leq 1000$, we see that

$$N-3 \leq \int_0^{1000} -3+6000/(1000+x)\,dx \leq N.$$

Therefore $N/60 \approx 50(2\ln 2 - 1)$. Using $\ln 2 \approx .69$, we conclude that it takes approximately 19 hours to complete the puzzle.

3. One can quickly check that $f(2) = 2$ and $f(3) = 3$, so it seems reasonable that $f(n) = n$, so let us try to prove this. Certainly if $f(n) = n$, then $f(1) = 1$, so we will prove the result by induction on n; we assume that the result is true for all integers $\leq n$. Then

$$f(n+1) = f(f(n)) + f(n+1-f(n)) = n + f(1) = n+1$$

as required and it follows that $f(n) = n$ for $n = 1, 2, \ldots$.

4. Write $P(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{Z}$. Let us suppose by way of contradiction that $a, b, c, d \geq -1$. From $P(2) = 0$, we get $8a + 4b + 2c + d = 0$, in particular d is even and hence $d \geq 0$. Since $4b + 2c + d \geq -7$, we see that $a \leq 0$. Also $a \neq 0$ because $P(x)$ has degree 3, so $a = -1$. We now have $4b + 2c + d = 8$ and $b + c + d = 1$ from $P(1) = 0$. Thus $-2c - 3d = 4$, so $-2c = 4 + 3d \geq 4$ and we conclude that $c \leq -2$. The result follows.

5. (a) For small positive x, we have $x/2 < \sin x < x$, so for positive integers n, we have $1/(2n) < \sin(1/n) < 1/n$. Since $\sum_{n=1}^{\infty} 1/n^p$ is convergent if and only if $p > 1$, it follows from the basic comparison test that $\sum_{n=1}^{\infty} (\sin 1/n)^p$ is convergent if and only if $p > 1$.

   (b) It is not difficult to show that any real number x, there exists an integer $n > x$ such that $|\sin n| > 1/2$. Thus whatever p is, $\lim_{n\to\infty} |\sin n|^p \neq 0$. Therefore $\sum_{n=1}^{\infty} |\sin n|^p$ is divergent for all p.

6. (a) If y^* is a steady-state solution, then $y^* = y^*(2 - y^*)$, so $y^* = 0$ or $1 = 2 - y^*$. Therefore the steady-state solutions are $y^* = 0$ or 1.

   (b) Suppose $0 < y_n < 1$. Then $y_{n+1}/y_n = 2 - y_n > 1$, so $y_{n+1} > y_n$. Also $y_{n+1} = 1 - (1 - y_n)^2$, so $y_{n+1} < 1$. We deduce that y_n is a monotonic positive increasing function that is bounded above by 1, in particular y_n converges to some positive number ≤ 1. It follows that y_n converges to 1.

7. Let $y \in [0, 1]$ be such that $(g(y) + uf(y)) = u$. Let us suppose we do have constants A and B such that $F(x) = Ag(x)/(f(x) + B)$ is a continuous function on $[0.1]$ with $\max_{0 \leq x \leq 1} F(x) = u$. We will guess that the maximum occurs when $x = y$, so $u = Ag(y)/(f(y) + B)$. Then $A = B = -1$ satisfies these equations, so $F(x) = g(x)/(1 - f(x))$.

   So let us prove that $F(x) = g(x)/(1 - f(x))$ has the required properties. Certainly $F(x)$ is continuous because $f(x) < 1$ for all $x \in [0, 1]$, and $F(y) = u$ from above. Finally $\max_{0 \leq x \leq 1}(g(x) + uf(x)) = u$, so $g(x) \leq u(1 - f(x))$ for all x and we conclude that $F(x) \leq u$. The result is proven.

8. Suppose we can disconnect F by removing only 8 points. Then the resulting framework will consist of two nonempty frameworks A, B such that there is no segment joining a point of A to a point of B. Let a be the number of points in A. Then there are $9 - a$ points in B, at most $a(a - 1)/2$ line segments joining the points of A, and at most $(10 - a)(10 - a - 1)/2$ line segments joining the points of B. It follows that the resulting framework has at most $45 - 10a + a^2$. Since $10a - a^2 > 8$ for $1 \leq a \leq 9$, the result follows.