11th VTRMC, 1989, Solutions

1. Let B be the area of the triangle, let A be the area of the top triangle, and set $x = b - a$. Then $B/A = (b/a)^2$, so $(B-A)/B = 1-(a/b)^2$. Since $B - A = a^2 + ax/2$, we see that $B = ab^2/(2x)$. Therefore $B - 2a^2 = (ab^2 - 4a^2x)(2x) = a(a-x)^2/(2a) \geq 0$. This proves the result.

2. It is easily checked that a 2×2 matrix with all elements 0 or 1 has determinant 0, or 1, or -1.

   (a) Suppose A has determinant ± 3. Then by expanding by the first row, we see that all entries of the first row must be 1. Similarly by expanding by the second row, we see that all entries of the second row must be 1. But this tells us that the determinant of A is 0 and the result follows.

   (b) From (a) and expanding by the first row, we now see that $\det A = 0, \pm 1, \pm 2$. It is very easy to see that $\det A$ can take the values 0 and ± 1. To get the value 2, set $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, while to get the value -2, take the above matrix and interchange the first two columns. Thus the values the determinant of A can take are precisely 0, ± 1, ± 2.

3. Note that $2(1,0,1)+(2,-2,1) = (2,-1,3)$. Therefore a solution of the system when $b_1 = 2$, $b_2 = -1$, $b_3 = 3$ is $(x_1,x_2,x_3) = 2(-1,3,2)+(2,-2,1)$. This yields the solution $x_1 = 0$, $x_2 = 4$, $x_3 = 5$.

4. We have $(r-a)(r-b)(r-c)(r-d) = 9$, where $r-a$, $r-b$, $r-c$, $r-d$ are distinct integers. This means that in some order, these four numbers must take the values ± 1, ± 3, in particular $(r-a)+(r-b)+(r-c)+(r-d) = 1-1+3-3 = 0$. The result follows.

5. (i) $1+x+x^2+x^3+x^4 = (x^5-1)(x-1)$. The only real root of x^5-1 is $x = 1$. The result follows.

   (ii) As indicated in the hint, we have $d/dx(xf_n(x)) = f_{n-1}(x)$. By part (i), we can assume that $f_{n-1}(x)$ has no real zero by induction on n. However it is then clear that $f_{n-1}(x)$ is always positive and therefore $xf_n(x)$ is a strictly increasing function. We deduce that $xf_n(x)$ has only one zero, namely $x = 0$, and the result follows.

6. Since
$$f(x/(x-1)) = \frac{x/(x-1)}{x/(x-1)-1} = \frac{x}{x-(x-1)} = x,$$
we see that $f^n(x) = x$ for x even and $f^n(x) = x/(x-1)$ for x odd. Therefore $\sum_{k=0}^{\infty} 2^{-k} f^k(x) = 4f^0(x)/3 + 2f^1(x)/3 = (4x^2-2x)/(3(x-1))$.

7. Let \$$x$ be the selling price before noon, and let \$$y$ be the selling price after noon. Let the first farmer sell a chickens before noon, the second farmer b chickens before noon, and the third farmer c chickens before noon. Then we have
$$\begin{aligned} ax + (10-a)y &= 35 \\ bx + (16-b)y &= 35 \\ cx + (26-c)y &= 35 \end{aligned}$$
Thus in particular
$$\begin{vmatrix} a & 10-a & 1 \\ b & 16-b & 1 \\ c & 26-c & 1 \end{vmatrix} = 0.$$
Remember that a, b, c are positive integers, $a < 10$, $b < 16$, $c < 26$. Also $x > y > 0$. By inspection, we must have $a = 9$, $b = 8$ and $c = 9$. This yields $x = 15/4$ and $y = 5/4$. Thus the cost of a chicken before noon is \$15/4, and the cost after noon is \$5/4.

8. The number of numbers in the sequence is the number of zero's, plus the number of 1's, plus the number of two's, plus .... In other words $n = a_0 + a_1 + \ldots a_{n-1}$. Also $a_0 \neq 0$, because if $a_0 = 0$, then since $n = a_0 + \cdots + a_n$, we would obtain $a_i = 1$ for all i. Thus there are $n - a_0 - 1$ nonzero terms in $\{a_1, \ldots, a_{n-1}\}$ which sum to $n - a_0$. Thus one of these nonzero terms is 2 and the rest are 1. If $a_1 = 0$, then $a_2 = 2$, hence $a_0 = 2$ and we have a contradiction because $n \geq 6$. If $a_1 = 1$, then $a_2 = 2$, hence $a_0 = 2$ and again we have a contradiction because $n \geq 6$. We deduce that $a_1 = 2$ and hence $a_2 = 1$. We conclude that the sequence must be $n-4, 2, 1, 0, \ldots, 0, 1, 0, 0, 0$. In particular for $n = 7$, the sequence is 3,2,1,1,0,0,0.