10th VTRMC, 1988, Solutions

1. Let $ABDE$ be the parallelogram $S$ and let the inscribed circle $C$ have center $O$. Thus $\angle AED = \theta$. Let $S$ touch $C$ at $P,Q,R,T$. It is well known that $S$ is a rhombus; to see this, note that $EP = ET, AP = AQ, BQ = BR$ and $DR = DT$. In particular $O$ is the intersection of $AD$ and $EB$, $\angle EOD = \pi/2$ and $\angle OED = \theta/2$. Let $x = \text{area of } S$. Then $x = 4$ times area of $EOD$. Since $ED = ET + TD = r \cot \theta/2 + r \tan \theta/2$, we conclude that $x = 2r^2(\cot \theta/2 + \tan \theta/2)$.

2. Let the check be for $x$ dollars and $y$ cents, so the original check is for $100x + y$ cents. Then $100y + x - 5 = 2(100x + y)$. Therefore $98y - 196x = 5 + 3x$. Of course, $x$ and $y$ are integers, and presumably $0 \leq x, y \leq 99$. Since 98 divides $5 + 3x$, we see that $x = 31$ and hence $y = 2x + (5 + 3x)/98 = 63$. Thus the original check was for $31.63$.

3. If we differentiate $y(x) + \int_1^x y(t) \, dt = x^2$ with respect to $x$, we obtain $y' + y = 2x$. This is a first order linear differential equations, and the general solution is $y = Ce^{-x} + 2x - 2$, where $C$ is an arbitrary constant. However when $x = 1$, $y(1) = 1$, so $1 = C/e + 2 - 2$ and hence $C = e$. Therefore $y = 2x - 2 + e^{1-x}$.

4. If $a = 1$, then $a^2 + b^2 = 2 = ab + 1$ for all $n$, and we see that $a^2 + b^2$ is always divisible by $ab + 1$. From now on, we assume that $n \geq 2$.

Suppose $a^{n+1} + 1$ divides $a^2 + a^{2n}$, where $n$ is a positive integer. Then $a^{n+1} + 1$ divides $a^{n+1} - a^2$ and hence $a^{n+1} + 1$ divides $a^4 + 1$. Thus in particular $n \leq 3$. If $n = 3$, then $a^{n+1} + 1 = a^4 + 1$ divides $a^2 + a^{2n} = a^2 + a^6$. If $n = 1$, then $a^2 + 1$ divides $2a^2$ implies $a^2 + 1$ divides $2$, which is not possible. Finally if $n = 2$, we obtain $a^3 + 1$ divides $a^2 + a^4$, so $a^3 + 1$ divides $a^2 - a$ which again is not possible.
8. If we have a triangle with integer sides $a, b, c$, then we obtain a triangle with integer sides $a+1, b+1, c+1$. Conversely if we have a triangle with integer sides $a, b, c$ and the perimeter $a+b+c$ is even, then none of $a, b, c$ can be 1 (because in a triangle, the sum of the lengths of any two sides is strictly greater than the length of the third side). This means we can obtain a triangle with sides $a-1, b-1, c-1$. We conclude that $T(n) = T(n-3)$ if $n$ is even.

5. Using Rolle’s theorem, we see that $f$ is either strictly monotonic increasing or strictly monotonic decreasing; without loss of generality assume that $f$ is monotonic increasing. Then $f'(X) \geq 2$ for all $x$, so $|x - x_0| < .00005$. Thus the smallest upper bound is .00005.

6. $f(x) = ax - bx^3$ has an extrema when $f'(x) = 0$, that is $a - 3bx^2 = 0$, so $x = \pm a/(\sqrt{3}b)$. Then $f(x) = \pm \frac{2a^3}{3\sqrt{3}b}$. Since $f$ has 4 extrema on $[-1, 1]$, two of the extrema must occur at $\pm 1$. Thus we have $|a-b| = 1$. Thus a possible choice is $a = .1$ and $b = 1.1$.

7. (a) $f([0, 1]) = [0, 1/3] \cup [2/3, 1]$

$$f(f([0, 1])) = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$


(b) Let $T \subseteq \mathbb{R}$ be a bounded set such that $f(T) = T$. First note that $T$ contains no negative numbers. Indeed if $T$ contains negative numbers, let $t_0 = \inf_{t \in T} t$ and choose $t_1 \in T$ with $t_1 < t_0/3$. Then there is no $t \in T$ with $f(t) = t_1$.

Therefore we may assume that $T$ contains no negative numbers. Now suppose $1/2 \in T$. Since $(x + 2)/3 = 1/2$ implies $x = -1/6$, we see that $3/2 \in T$. Now let $t_2 = \sup_{t \in T} t$ and choose $t_3 \in T$ such that $t_3 > (t_2 + 2)/3$. Since $f(T) = T$, we see that there exists $s \in T$ such that $f(s) > (t_2 + 2)/3$, which is not possible.

We conclude that there is no bounded subset $T$ such that $f(T) = T$ and $1/2 \in T$.

8. If we have a triangle with integer sides $a, b, c$, then we obtain a triangle with integer sides $a+1, b+1, c+1$. Conversely if we have a triangle with integer sides $a, b, c$ and the perimeter $a+b+c$ is even, then none of $a, b, c$ can be 1 (because in a triangle, the sum of the lengths of any two sides is strictly greater than the length of the third side). This means we can obtain a triangle with sides $a-1, b-1, c-1$. We conclude that $T(n) = T(n-3)$ if $n$ is even.