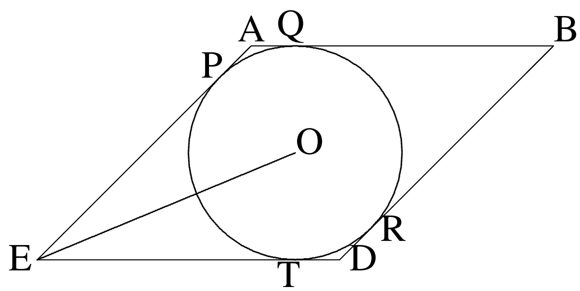
10th VTRMC, 1988, Solutions

1. Let $ABDE$ be the parallelogram S and let the inscribed circle C have center O. Thus $\angle AED = \theta$. Let S touch C at P, Q, R, T. It is well known that S is a rhombus; to see this, note that $EP = ET$, $AP = AQ$, $BQ = BR$ and $DR = DT$. In particular O is the intersection of AD and EB, $\angle EOD = \pi/2$ and $\angle OED = \theta/2$. Let $x =$ area of S. Then $x = 4$ times area of EOD. Since $ED = ET + TD = r\cot\theta/2 + r\tan\theta/2$, we conclude that $x = 2r^2(\cot\theta/2 + \tan\theta/2)$.



2. Let the check be for x dollars and y cents, so the original check is for $100x + y$ cents. Then $100y + x - 5 = 2(100x + y)$. Therefore $98y - 196x = 5 + 3x$. Of course, x and y are integers, and presumably $0 \le x, y \le 99$. Since 98 divides $5 + 3x$, we see that $x = 31$ and hence $y = 2x + (5 + 3x)/98 = 63$. Thus the original check was for \$31.63.

3. If we differentiate $y(x) + \int_1^x y(t)\,dt = x^2$ with respect to x, we obtain $y' + y = 2x$. This is a first order linear differential equations, and the general solution is $y = Ce^{-x} + 2x - 2$, where C is an arbitrary constant. However when $x = 1$, $y(1) = 1$, so $1 = C/e + 2 - 2$ and hence $C = e$. Therefore $y = 2x - 2 + e^{1-x}$.

4. If $a = 1$, then $a^2 + b^2 = 2 = ab + 1$ for all n, and we see that $a^2 + b^2$ is always divisible by $ab + 1$. From now on, we assume that $n \ge 2$.

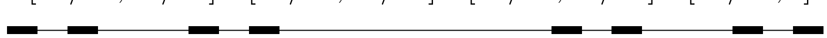
   Suppose $a^{n+1} + 1$ divides $a^2 + a^{2n}$, where n is a positive integer. Then $a^{n+1} + 1$ divides $a^{n-1} - a^2$ and hence $a^{n+1} + 1$ divides $a^4 + 1$. Thus in particular $n \le 3$. If $n = 3$, then $a^{n+1} + 1 = a^4 + 1$ divides $a^2 + a^{2n} = a^2 + a^6$. If $n = 1$, then $a^2 + 1$ divides $2a^2$ implies $a^2 + 1$ divides 2, which is not possible. Finally if $n = 2$, we obtain $a^3 + 1$ divides $a^2 + a^4$, so $a^3 + 1$ divides $a^2 - a$ which again is not possible.

We conclude that if $a > 1$, then $a^2 + b^2$ is divisible by $ab + 1$ if and only if $n = 3$.

5. Using Rolle's theorem, we see that f is either strictly monotonic increasing or strictly monotonic decreasing; without loss of generality assume that f is monotonic increasing. Then $f'(X) \geq 2$ for all x, so $|\alpha - x_0| < .00005$. Thus the smallest upper bound is .00005.

6. $f(x) = ax - bx^3$ has an extrema when $f'(x) = 0$, that is $a - 3bx^2 = 0$, so $x = \pm a/(\sqrt{3}b)$. Then $f(x) = \pm\frac{2a^2}{3\sqrt{3b}}$. Since f has 4 extrema on $[-1, 1]$, two of the extrema must occur at ± 1. Thus we have $|a - b| = 1$. Thus a possible choice is $a = .1$ and $b = 1.1$.

7. (a)
$$\begin{aligned} f([0,1]) &= [0,1/3] \cup [2/3,1] \\ f(f([0,1])) &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1] \\ f(f(f([0,1]))) &= [0,1/27] \cup [2/27,3/27] \cup [6/27,7,27] \cup [8/27,9/27] \cup \\ & [18/27,19/27] \cup [20/27,21/27] \cup [24/27,25/27] \cup [26/27,1] \end{aligned}$$



   (b) Let $T \subseteq \mathbb{R}$ be a bounded set such that $f(T) = T$. First note that T contains no negative numbers. Indeed if T contains negative numbers, let $t_0 = \inf_{t \in T} t$ and choose $t_1 \in T$ with $t_1 < t_0/3$. Then there is no $t \in T$ with $f(t) = t_1$.

   Therefore we may assume that T contains no negative numbers. Now suppose $1/2 \in T$. Since $(x+2)/3 = 1/2$ implies $x = -1/6$, we see that $3/2 \in T$. Now let $t_2 = \sup_{t \in T} t$ and choose $t_3 \in T$ such that $t_3 > (t_2 + 2)/3$. Since $f(T) = T$, we see that there exists $s \in T$ such that $f(s) > (t_2 + 2)/3$, which is not possible.

   We conclude that there is no bounded subset T such that $f(T) = T$ and $1/2 \in T$.

8. If we have a triangle with integer sides a, b, c, then we obtain a triangle with integer sides $a+1, b+1, c+1$. Conversely if we have a triangle with integer sides a, b, c and the perimeter $a + b + c$ is even, then none of a, b, c can be 1 (because in a triangle, the sum of the lengths of any two sides is strictly greater than the length of the third side). This means we can obtain a triangle with sides $a-1, b-1, c-1$. We conclude that $T(n) = T(n-3)$ if n is even.