9th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 31, 1987

Fill out the individual registration form

1. A path zig-zags from $(1,0)$ to $(0,0)$ along line segments $\overline{P_nP_{n+1}}$, where P_0 is $(1,0)$ and P_n is $(2^{-n},(-2)^{-n})$, for $n>0$. Find the length of the path.

2. A triangle with sides of lengths a, b, and c is partitioned into two smaller triangles by the line which is perpendicular to the side of length c and passes through the vertex opposite that side. Find *integers* $a<b<c$ such that each of the two smaller triangles is similar to the original triangle and has sides of integer lengths.

3. Let $a_1,a_2,\ldots,a_n$ be an arbitrary rearrangement of $1,2,\ldots,n$. Prove that if n is odd, then $(a_1-1)(a_2-2)\ldots(a_n-n)$ is even.

4. Let $p(x)$ be given by $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and let $|p(x)|\leq|x|$ on $[-1,1]$.

   (a) Evaluate a_0. (b) Prove that $|a_1|\leq 1$.

5. A sequence of integers $\{n_1,n_2,\ldots\}$ is defined as follows: n_1 is assigned arbitrarily and, for $k>1$,

$$n_k=\sum_{j=1}^{j=k-1} z(n_j),$$

   where $z(n)$ is the number of 0's in the binary representation of n (each representation should have a leading digit of 1 except for zero which has the representation 0). An example, with $n_1=9$, is $\{9,2,3,3,3,\ldots\}$, or in binary, $\{1001,10,11,11,11,\ldots\}$.

   (a) Find n_1 so hat $\lim_{k\to\infty} n_k=31$, and calculate $n_2,n_3,\ldots,n_{10}$.

   (b) Prove that, for every choice of n_1, the sequence $\{n_k\}$ converges.

6. A sequence of polynomials is given by $p_n(x)=a_{n+2}x^2+a_{n+1}x-a_n$, for $n\geq 0$, where $a_0=a_1=1$ and, for $n\geq 0$, $a_{n+2}=a_{n+1}+a_n$. Denote by r_n and s_n the roots of $p_n(x)=0$, with $r_n\leq s_n$. Find $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} s_n$.

7. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices such that A^{-1} exists. Define $A(t) = \{a_{ij}(t)\}$ and $B(t) = \{b_{ij}(t)\}$ by $a_{ij}(t) = a_{ij}$ for $i < n$, $a_{nj}(t) = ta_{nj}$, $b_{ij}(t) = b_{ij}$ for $i < n$, and $b_{nj}(t) = tb_{nj}$. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A(t) = \begin{bmatrix} 1 & 2 \\ 3t & 4t \end{bmatrix}$. Prove that $A(t)^{-1}B(t) = A^{-1}B$ for $t > 0$ and any n. (Partial credit will be given for verifying the result for $n = 3$.)

8. On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y-axis. The witch is *invisible except by reflection* in the mirror. At $t = 0$, the cat is at $(10, 10)$ and the witch is at $(10, 0)$. For $t \geq 0$, the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by $(u(t), v(t))$ the position of the cat and by $(x(t), y(t))$ the position of the witch.

   (a) Set up the equations of motion of the cat and the witch for $t \geq 0$.

   (b) Solve for $x(t)$ and $u(t)$ and find the time when the cat strikes the mirror. (Recall that the mirror is a perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)