8th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, November 1, 1986

Fill out the individual registration form

1. Let $x_1 = 1, x_2 = 3$, and
$$x_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n} x_i \quad \text{for } n = 2, 3, \ldots.$$
Find $\lim_{n\to\infty}$ and give a proof of your answer.

2. Given that $a > 0$ and $c > 0$, find a necessary and sufficient condition on b so that $ax^2 + bx + c > 0$ for all $x > 0$.

3. Express $\sinh 3x$ as a polynomial in $\sinh x$. As an example, the identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be expressed as a polynomial in $\cos x$. (Recall that sinh denotes the hyperbolic sine defined by $\sinh x = (e^x - e^{-x})/2$.)

4. Find the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that $\int_0^1 t^n p(t)\,dt = n$ for $n = 0, 1, 2$.

5. Verify that, for $f(x) = x + 1$,
$$\lim_{r\to 0^+} \left(\int_0^1 (f(x))^r\,dx\right)^{1/r} = e^{\int_0^1 \ln f(x)\,dx}.$$

6. Sets A and B are defined by $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, 3\}$. Determine the number of distinct functions from A onto B. (A function $f\colon A \to B$ is "onto" if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.)

7. A function f from the positive integers to the positive integers has the properties:
   - $f(1) = 1$,
   - $f(n) = 2$ if $n \geq 100$,
   - $f(n) = f(n/2)$ if n is even and $n < 100$,

- $f(n) = f(n^2+7)$ if n is odd and $n > 1$.

(a) Find all positive integers n for which the stated properties require that $f(n) = 1$.

(b) Find all positive integers n for which the stated properties do not determine $f(n)$.

8. Find all pairs N, M of positive integers, $N < M$, such that

$$\sum_{j=N}^{M} \frac{1}{j(j+1)} = \frac{1}{10}.$$