4th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 6, 1982

Fill out the individual registration form

1. What is the remainder when $X^{1982}+1$ is divided by $X-1$? Verify your answer.

2. A box contains marbles, each of which is red, white or blue. The number of blue marbles is a least half the number of white marbles and at most on-third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.

3. Let **a**, **b**, and **c** be vectors such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is linearly dependent. Show that

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = 0.$$

4. Prove that $t^{n-1}+t^{1-n}<t^n+t^{-n}$ when $t\neq 1, t>0$ and n is a positive integer.

5. When asked to state the Maclaurin Series, a student writes (incorrectly)

$$(*) \quad f(x) = f(x) + xf'(x) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots.$$

   (a) State Maclaurin's Series for $f(x)$ correctly.

   (b) Replace the left-hand side of (*) by a simple closed form expression in f in such a way that the statement becomes valid (in general).

6. Let S be a set of positive integers and let E be the operation on the set of subsets of S defined by $EA=\{x\in A \mid x \text{ is even}\}$, where $A\subseteq S$. Let CA denote the complement of A in S. $ECEA$ will denote $E(C(EA))$ etc.

   (a) Show that $ECECEA = EA$.

   (b) Find the maximum number of distinct subsets of S that can be generated by applying the operations E and C to a subset A of S an arbitrary number of times in any order.

7. Let $p(x)$ be a polynomial of the form $p(x) = ax^2 + bx + c$, where a, b and c are *integers*, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \geq 0$.

8. For $n \geq 2$, define S_n by $S_n = \sum_{k=n}^{\infty} \frac{1}{k^2}$.

   (a) Prove or disprove that $1/n < S_n < 1/(n-1)$.

   (b) Prove or disprove that $S_n < 1/(n - 3/4)$.