41st Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 26, 2019

Fill out the individual registration form

1. For each positive integer n, define $f(n)$ to be the sum of the digits of 2771^n (so $f(1) = 17$). Find the minimum value of $f(n)$ (where $n \geq 1$). Justify your answer.

2. Let X be the point on the side AB of the triangle ABC such that $BX/XA = 9$. Let M be the midpoint of BX and let Y be the point on BC such that $\angle BMY = 90^\circ$. Suppose AC has length 20 and that the area of the triangle XYC is 9/100 of the area of the triangle ABC. Find the length of BC.

3. Let n be a nonnegative integer and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial with real coefficients a_i. Suppose that

$$\frac{a_n}{(n+1)(n+2)} + \frac{a_{n-1}}{n(n+1)} + \cdots + \frac{a_1}{6} + \frac{a_0}{2} = 0.$$

   Prove that $f(x)$ has a real zero.

4. Compute $\displaystyle\int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx$ (the answer is a rational number).

5. Find the general solution of the differential equation

$$x^4 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + (1-2x)y = 0$$

   valid for $0 < x < \infty$.

6. Let S be a subset of $\mathbb{R}$ with the property that for every $s \in S$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap S = \{s\}$. Prove there exists a function $f\colon S \to \mathbb{N}$, the positive integers, such that for all $s,t \in S$, if $s \neq t$ then $f(s) \neq f(t)$.

7. Let S denote the positive integers that have no 0 in their decimal expansion. Determine whether $\displaystyle\sum_{n\in S} n^{-99/100}$ is convergent.