40th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2018

Fill out the individual registration form

1. It is known that $\int_1^2 \frac{\arctan(1+x)}{x}\,dx = q\pi\ln(2)$ for some rational number q. Determine q. Here $0 \le \arctan(x) < \pi/2$ for $0 \le x < \infty$.

2. Let $A, B \in \mathrm{M}_6(\mathbb{Z})$ such that $A \equiv I \equiv B \bmod 3$ and $A^3B^3A^3 = B^3$. Prove that $A = I$. Here $\mathrm{M}_6(\mathbb{Z})$ indicates the 6 by 6 matrices with integer entries, I is the identity matrix, and $X \equiv Y \bmod 3$ means all entries of $X - Y$ are divisible by 3.

3. Prove that there is no function $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n+1$. Here $\mathbb{N}$ is the positive integers $\{1, 2, 3, \dots\}$.

4. Let m, n be integers such that $n \ge m \ge 1$. Prove that $\frac{\gcd(m,n)}{n}\binom{n}{m}$ is an integer. Here gcd denotes greatest common divisor and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

5. For $n \in \mathbb{N}$, let $a_n = \int_0^{1/\sqrt{n}} |1 + e^{it} + e^{2it} + \dots + e^{nit}|\,dt$. Determine whether the sequence $(a_n) = a_1, a_2, \dots$ is bounded.

6. For $n \in \mathbb{N}$, define $a_n = \frac{1 + 1/3 + 1/5 + \dots + 1/(2n-1)}{n+1}$ and $b_n = \frac{1/2 + 1/4 + 1/6 + \dots + 1/(2n)}{n}$. Find the maximum and minimum of $a_n - b_n$ for $1 \le n \le 999$.

7. A continuous function $f \colon [a,b] \to [a,b]$ is called piecewise monotone if $[a,b]$ can be subdivided into finitely many subintervals

$$I_1 = [c_0, c_1],\ I_2 = [c_1, c_2],\ \dots,\ I_\ell = [c_{\ell-1}, c_\ell]$$

such that f restricted to each interval I_j is strictly monotone, either increasing or decreasing. Here we are assuming that $a = c_0 < c_1 < \dots < c_{\ell-1} < c_\ell = b$. We are also assuming that each I_j is a maximal interval on which f is strictly monotone. Such a maximal interval is called a lap of the function

f, and the number $\ell = \ell(f)$ of distinct laps is called the lap number of f. If $f\colon [a,b] \to [a,b]$ is a continuous piecewise-monotone function, show that the sequence $(\sqrt[n]{\ell(f^n)})$ converges; here f^n means f composed with itself n-times, so $f^2(x) = f(f(x))$ etc.