1. The number $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?

2. For which real numbers $b$ does the function $f(x)$, defined by the conditions $f(0) = b$ and $f'' = 2f - x$, satisfy $f(x) > 0$ for all $x \geq 0$?

3. Let $A$ be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with $A$ being otherwise arbitrary.
   (a) Express $(I - A)^{-1}$ as a polynomial in $A$, where $I$ is the identity matrix.
   (b) Find a $3 \times 3$ matrix satisfying $B^2 \neq 0, B^3 = 0$.

4. Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where $F_n$ is the $n$th Fibonacci number defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$, $n > 1$. Find an explicit closed form for $F(x)$.

5. Two elements $A, B$ in a group $G$ have the property $ABA^{-1}B = 1$, where 1 denotes the identity element in $G$.
   (a) Show that $AB^2 = B^{-2}A$.
   (b) Show that $AB^n = B^{-n}A$ for any integer $n$.
   (c) Find $u$ and $v$ so that $(B^u A^b)(B^v A^d) = B^u A^v$.

6. With $k$ a positive integer, prove that $(1 - k^{-2})^k \geq 1 - 1/k$.

7. Let $A = \{a_0, a_1, \ldots \}$ be a sequence of real numbers and define the sequence $A' = \{a'_0, a'_1, \ldots \}$ as follows for $n = 0, 1, \ldots$: $a'_{2n} = a_n, a'_{2n+1} = a_{n+1}$. If $a_0 = 1$ and $A' = A$, find
   (a) $a_1, a_2, a_3$ and $a_4$
   (b) $a_{1981}$
   (c) A simple general algorithm for evaluating $a_n$, for $n = 0, 1, \ldots$

8. Let
(i) $0 < a < 1$,
(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, \ldots$,
(iii) $\lim_{k \to \infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n \to \infty} b_n = 0$. 