# 38th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 22, 2016

## Fill out the individual registration form

1. Evaluate $\displaystyle\int_1^2 \frac{\ln x}{2-2x+x^2}\,dx$.

2. Determine the real numbers $k$ such that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(2n)!}{4^n n!n!}\right)^k$ is convergent.

3. Let $n$ be a positive integer and let $\mathrm{M}_n(\mathbb{Z}_2)$ denote the $n$ by $n$ matrices with entries from the integers mod 2. If $n \geq 2$, prove that the number of matrices $A$ in $\mathrm{M}_n(\mathbb{Z}_2)$ satisfying $A^2 = 0$ (the matrix with all entries zero) is an even positive integer.

4. For a positive integer $a$, let $P(a)$ denote the largest prime divisor of $a^2 + 1$. Prove that there exist infinitely many triples $(a,b,c)$ of distinct positive integers such that $P(a) = P(b) = P(c)$.

5. Suppose that $m,n,r$ are positive integers such that
$$1+m+n\sqrt{3} = (2+\sqrt{3})^{2r-1}.$$
Prove that $m$ is a perfect square.

6. Let $A,B,P,Q,X,Y$ be square matrices of the same size. Suppose that
$$A+B+AB = XY \qquad AX = XQ$$
$$P+Q+PQ = YX \qquad PY = YB.$$
Prove that $AB = BA$.

7. Let $q$ be a real number with $|q| \neq 1$ and let $k$ be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable $X$, depending on $q$ and $k$, by $f_k(X) = \displaystyle\prod_{i=0}^{k-1}(1-q^iX)(1-q^{i+1}X^{-1})$. (Here $\prod$ denotes product.) Show that the constant term of $f_k(X)$, i.e. the coefficient of $X^0$ in $f_k(X)$, is equal to
$$\frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{2k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$