36th Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 25, 2014

Fill out the individual registration form

1. Find \( \sum_{n=2}^{n=\infty} \frac{n^2 - 2n - 4}{n^4 + 4n^2 + 16}. \)

2. Evaluate \( \int_0^2 \frac{(16 - x^2)x}{16 - x^2 + \sqrt{(4 - x)(4 + x)(12 + x^2)}} \, dx. \)

3. Find the least positive integer \( n \) such that \( 2^{2014} \) divides \( 19^n - 1. \)

4. Suppose we are given a \( 19 \times 19 \) chessboard (a table with \( 19^2 \) squares) and remove the central square. Is it possible to tile the remaining \( 19^2 - 1 = 360 \) squares with \( 4 \times 1 \) and \( 1 \times 4 \) rectangles? (So each of the 360 squares is covered by exactly one rectangle.) Justify your answer.

5. Let \( n \geq 1 \) and \( r \geq 2 \) be positive integers. Prove that there is no integer \( m \) such that \( n(n + 1)(n + 2) = m^r. \)

6. Let \( S \) denote the set of \( 2 \) by \( 2 \) matrices with integer entries and determinant \( 1 \), and let \( T \) denote those matrices of \( S \) which are congruent to the identity matrix \( I \mod 3 \) (so \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T \) means that \( a, b, c, d \in \mathbb{Z}, ad - bc = 1, \) and 3 divides \( b, c, a - 1, d - 1; \) “\( \in \)” means “is in”).

   (a) Let \( f : T \rightarrow \mathbb{R} \) (the real numbers) be a function such that for every \( X, Y \in T \) with \( Y \neq I \), either \( f(XY) > f(X) \) or \( f(XY^{-1}) > f(X) \) (or both). Show that given two finite nonempty subsets \( A, B \) of \( T \), there are matrices \( a \in A \) and \( b \in B \) such that if \( a' \in A, b' \in B \) and \( a' b' = ab \), then \( a' = a \) and \( b' = b. \)

   (b) Show that there is no \( f : S \rightarrow \mathbb{R} \) such that for every \( X, Y \in S \) with \( Y \neq \pm I \), either \( f(XY) > f(X) \) or \( f(XY^{-1}) > f(X) \).

(Please turn over)
7. Let $A, B$ be two points in the plane with integer coordinates $A = (x_1, y_1)$ and $B = (x_2, y_2)$. (Thus $x_i, y_i \in \mathbb{Z}$, for $i = 1, 2$.) A path $\pi: A \rightarrow B$ is a sequence of down and right steps, where each step has an integer length, and the initial step starts from $A$, the last step ending at $B$. In the figure below, we indicated a path from $A_1 = (4, 9)$ to $B_1 = (10, 3)$. The distance $d(A, B)$ between $A$ and $B$ is the number of such paths. For example, the distance between $A = (0, 2)$ and $B = (2, 0)$ equals 6. Consider now two pairs of points in the plane $A_i = (x_i, y_i)$ and $B_i = (u_i, z_i)$ for $i = 1, 2$, with integer coordinates, and in the configuration shown in the picture (but with arbitrary coordinates):

- $x_2 < x_1$ and $y_1 > y_2$, which means that $A_1$ is North-East of $A_2$; $u_2 < u_1$ and $z_1 > z_2$, which means that $B_1$ is North-East of $B_2$.
- Each of the points $A_i$ is North-West of the points $B_j$, for $1 \leq i, j \leq 2$. In terms of inequalities, this means that $x_i < \min\{u_1, u_2\}$ and $y_i > \max\{z_1, z_2\}$ for $i = 1, 2$.

(a) Find the distance between two points $A$ and $B$ as before, as a function of the coordinates of $A$ and $B$. Assume that $A$ is North-West of $B$.

(b) Consider the $2 \times 2$ matrix $M = \begin{pmatrix} d(A_1, B_1) & d(A_1, B_2) \\ d(A_2, B_1) & d(A_2, B_2) \end{pmatrix}$. Prove that for any configuration of points $A_1, A_2, B_1, B_2$ as described before, $\det M > 0$. 

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