36th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 25, 2014

Fill out the individual registration form

1. Find $\displaystyle\sum_{n=2}^{n=\infty} \frac{n^2-2n-4}{n^4+4n^2+16}$.

2. Evaluate $\displaystyle\int_0^2 \frac{(16-x^2)x}{16-x^2+\sqrt{(4-x)(4+x)(12+x^2)}}\,dx$.

3. Find the least positive integer n such that 2^{2014} divides 19^n-1.

4. Suppose we are given a 19×19 chessboard (a table with 19^2 squares) and remove the central square. Is it possible to tile the remaining $19^2-1=360$ squares with 4×1 and 1×4 rectangles? (So each of the 360 squares is covered by exactly one rectangle.) Justify your answer.

5. Let $n\geq 1$ and $r\geq 2$ be positive integers. Prove that there is no integer m such that $n(n+1)(n+2)=m^r$.

6. Let S denote the set of 2 by 2 matrices with integer entries and determinant 1, and let T denote those matrices of S which are congruent to the identity matrix I mod 3 (so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$ means that $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$, and 3 divides b, c, $a-1$, $d-1$; "$\in$" means "is in").

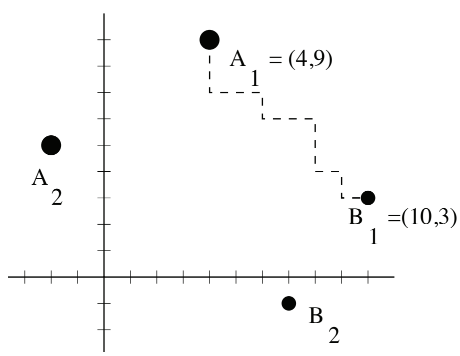
   (a) Let $f\colon T\to\mathbb{R}$ (the real numbers) be a function such that for every $X,Y\in T$ with $Y\neq I$, either $f(XY)>f(X)$ or $f(XY^{-1})>f(X)$ (or both). Show that given two finite nonempty subsets A,B of T, there are matrices $a\in A$ and $b\in B$ such that if $a'\in A$, $b'\in B$ and $a'b'=ab$, then $a'=a$ and $b'=b$.

   (b) Show that there is no $f\colon S\to\mathbb{R}$ such that for every $X,Y\in S$ with $Y\neq\pm I$, either $f(XY)>f(X)$ or $f(XY^{-1})>f(X)$.

(Please turn over)

7. Let A, B be two points in the plane with integer coordinates $A = (x_1, y_1)$ and $B = (x_2, y_2)$. (Thus $x_i, y_i \in \mathbb{Z}$, for $i = 1, 2$.) A path $\pi \colon A \to B$ is a sequence of **down** and **right** steps, where each step has an integer length, and the initial step starts from A, the last step ending at B. In the figure below, we indicated a path from $A_1 = (4, 9)$ to $B_1 = (10, 3)$. The distance $d(A, B)$ between A and B is the number of such paths. For example, the distance between $A = (0, 2)$ and $B = (2, 0)$ equals 6. Consider now two pairs of points in the plane $A_i = (x_i, y_i)$ and $B_i = (u_i, z_i)$ for $i = 1, 2$, with integer coordinates, and in the configuration shown in the picture (but with arbitrary coordinates):

- $x_2 < x_1$ and $y_1 > y_2$, which means that A_1 is North-East of A_2; $u_2 < u_1$ and $z_1 > z_2$, which means that B_1 is North-East of B_2.
- Each of the points A_i is North-West of the points B_j, for $1 \leq i, j \leq 2$. In terms of inequalities, this means that $x_i < \min\{u_1, u_2\}$ and $y_i > \max\{z_1, z_2\}$ for $i = 1, 2$.



(a) Find the distance between two points A and B as before, as a function of the coordinates of A and B. Assume that A is North-West of B.

(b) Consider the 2×2 matrix $M = \begin{pmatrix} d(A_1, B_1) & d(A_1, B_2) \\ d(A_2, B_1) & d(A_2, B_2) \end{pmatrix}$. Prove that for any configuration of points A_1, A_2, B_1, B_2 as described before, $\det M > 0$.