

**34th Annual Virginia Tech Regional Mathematics Contest**  
From 9:00 a.m. to 11:30 a.m., October 27, 2012

**Fill out the individual registration form**

1. Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2 \sin x \cos^3 x + 2 \sin^2 x \cos^2 x + 2 \sin^3 x \cos x} dx.$$

2. Solve in real numbers the equation  $3x - x^3 = \sqrt{x+2}$ .

3. Find nonzero complex numbers  $a, b, c, d, e$  such that

$$\begin{aligned} a + b + c + d + e &= -1 \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 15 \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} &= -1 \\ \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} &= 15 \\ abcde &= -1 \end{aligned}$$

4. Define  $f(n)$  for  $n$  a positive integer by  $f(1) = 3$  and  $f(n+1) = 3^{f(n)}$ . What are the last two digits of  $f(2012)$ ?

5. Determine whether the series  $\sum_{n=2}^{\infty} \frac{1}{\ln n} - \left(\frac{1}{\ln n}\right)^{(n+1)/n}$  is convergent.

6. Define a sequence  $(a_n)$  for  $n$  a positive integer inductively by  $a_1 = 1$  and  $a_n = \frac{n}{\prod_{1 \leq d|n, d < n} a_d}$ . Thus  $a_2 = 2, a_3 = 3, a_4 = 2$  etc. Find  $a_{999000}$ .

7. Let  $A_1, A_2, A_3$  be  $2 \times 2$  matrices with entries in  $\mathbb{C}$  (the complex numbers). Let  $\text{tr}$  denote the trace of a matrix (so  $\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$ ). Suppose  $\{A_1, A_2, A_3\}$  is closed under matrix multiplication (i.e. given  $i, j$ , there exists  $k$  such that  $A_i A_j = A_k$ ), and  $\text{tr}(A_1 + A_2 + A_3) \neq 3$ . Prove that there exists  $i$  such that  $A_i A_j = A_j A_i$  for all  $j$  (here  $i, j$  are 1, 2 or 3).