32nd Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 30, 2010

Fill out the individual registration form

1. Let \( d \) be a positive integer and let \( A \) be a \( d \times d \) matrix with integer entries. Suppose \( I + A + A^2 + \cdots + A^{100} = 0 \) (where \( I \) denotes the identity \( d \times d \) matrix, so \( I \) has 1’s on the main diagonal, and 0 denotes the zero matrix, which has all entries 0). Determine the positive integers \( n \leq 100 \) for which \( A^n + A^{n+1} + \cdots + A^{100} \) has determinant \( \pm 1 \).

2. For \( n \) a positive integer, define \( f_1(n) = n \) and then for \( i \) a positive integer, define \( f_{i+1}(n) = f_i(n)^{f_i(n)} \). Determine \( f_{100}(75) \mod 17 \) (i.e. determine the remainder after dividing \( f_{100}(75) \) by 17, an integer between 0 and 16). Justify your answer.

3. Prove that \( \cos(\pi/7) \) is a root of the equation \( 8x^3 - 4x^2 - 4x + 1 = 0 \), and find the other two roots.

4. Let \( \triangle ABC \) be a triangle with sides \( a, b, c \) and corresponding angles \( A, B, C \) (so \( a = BC \) and \( A = \angle BAC \) etc.). Suppose that \( 4A + 3C = 540^\circ \). Prove that \( (a - b)^2(a + b) = bc^2 \).

5. Let \( A, B \) be two circles in the plane with \( B \) inside \( A \). Assume that \( A \) has radius 3, \( B \) has radius 1, \( P \) is a point on \( A, Q \) is a point on \( B \), and \( A \) and \( B \) touch so that \( P \) and \( Q \) are the same point. Suppose that \( A \) is kept fixed and \( B \) is rolled once round the inside of \( A \) so that \( Q \) traces out a curve starting and finishing at \( P \). What is the area enclosed by this curve?

(Please turn over)
6. Define a sequence by \( a_1 = 1, \) \( a_2 = 1/2, \) and \( a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2} \) for \( n \) a positive integer. Find \( \lim_{n \to \infty} na_n. \)

7. Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series of positive terms (so \( a_i > 0 \) for all \( i \)) and set \( b_n = \frac{1}{na_n^2} \) for \( n \geq 1. \) Prove that \( \sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n} \) is convergent.