2nd Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 8, 1980

Fill out the individual registration form

1. Let * denote a binary operation on a set S with the property that
   \[(w*x)*(y*z) = w*z\]
   for all \(w,x,y,z \in S\).
   Show
   (a) If \(a*b = c\), then \(c*c = c\).
   (b) If \(a*b = c\), then \(a*x = c*x\) for all \(x \in S\).

2. The sum of the first \(n\) terms of the sequence
   
   \(1, (1+2), (1+2+2^2), \ldots, (1+2+\cdots+2^{k-1}), \ldots\)
   
   is of the form \(2^n+R+Sn^2+Tn+U\) for all \(n > 0\). Find \(R,S,T\) and \(U\).

3. Let \(a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots \cdot 2n}\).
   (a) Prove that \(\lim_{n \to \infty} a_n\) exists.
   (b) Show that \(a_n = \frac{(1 - \left(\frac{1}{2}\right)^2)(1 - \left(\frac{1}{3}\right)^2) \cdots (1 - \left(\frac{1}{n}\right)^2)}{(2n+1)a_n}\).
   (c) Find \(\lim_{n \to \infty} a_n\) and justify your answer.

4. Let \(P(x)\) be any polynomial of degree at most 3. It can be shown that there are numbers \(x_1\) and \(x_2\) such that \(\int_{-1}^{1} P(x) \, dx = P(x_1) + P(x_2)\), where \(x_1\) and \(x_2\) are independent of the polynomial \(P\).
   (a) Show that \(x_1 = -x_2\).
   (b) Find \(x_1\) and \(x_2\).

5. For \(x > 0\), show that \(e^x < (1+x)^{1+x}\).

6. Given the linear fractional transformation of \(x\) into \(f_1(x) = (2x-1)/(x+1)\), define \(f_{n+1}(x) = f_1(f_n(x))\) for \(n = 1, 2, 3, \ldots\). It can be shown that \(f_{35} = f_5\). Determine \(A, B, C,\) and \(D\) so that \(f_{28}(x) = (Ax+B)/(Cx+D)\).
7. Let $S$ be the set of all ordered pairs of integers $(m, n)$ satisfying $m > 0$ and $n < 0$. Let $\langle$ be a partial ordering on $S$ defined by the statement: $(m, n) \langle (m', n')$ if and only if $m \leq m'$ and $n \leq n'$. An example is $(5, -10) \langle (8, -2)$. Now let $O$ be a completely ordered subset of $S$, i.e. if $(a, b) \in O$ and $(c, d) \in O$, then $(a, b) \langle (c, d)$ or $(c, d) \langle (a, b)$. Also let $\mathcal{O}$ denote the collection of all such completely ordered sets.

(a) Determine whether an arbitrary $O \in \mathcal{O}$ is finite.

(b) Determine whether the cardinality $\|O\|$ of $O$ is bounded for $O \in \mathcal{O}$.

(c) Determine whether $\|O\|$ can be countably infinite for any $O \in \mathcal{O}$.

8. Let $z = x + iy$ be a complex number with $x$ and $y$ rational and with $|z| = 1$.

(a) Find two such complex numbers.

(b) Show that $|z^{2n} - 1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.

(c) Show that $|z^{2n} - 1|$ is rational for every $n$. 
