2nd Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 8, 1980

Fill out the individual registration form

1. Let * denote a binary operation on a set S with the property that

$$(w*x)*(y*z) = w*z \quad \text{for all } w,x,y,z \in S.$$

   Show

   (a) If $a*b = c$, then $c*c = c$.

   (b) If $a*b = c$, then $a*x = c*x$ for all $x \in S$.

2. The sum of the first n terms of the sequence

$$1, \quad (1+2), \quad (1+2+2^2), \ldots, (1+2+\cdots+2^{k-1}), \ldots$$

   is of the form $2^{n+R} + Sn^2 + Tn + U$ for all $n > 0$. Find R, S, T and U.

3. Let $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}$.

   (a) Prove that $\lim_{n\to\infty} a_n$ exists.

   (b) Show that $a_n = \dfrac{(1-(\frac{1}{2})^2)(1-(\frac{1}{4})^2)\ldots(1-(\frac{1}{2n})^2)}{(2n+1)a_n}$.

   (c) Find $\lim_{n\to\infty} a_n$ and justify your answer.

4. Let $P(x)$ be any polynomial of degree at most 3. It can be shown that there are numbers x_1 and x_2 such that $\int_{-1}^{1} P(x)\,dx = P(x_1) + P(x_2)$, where x_1 and x_2 are independent of the polynomial P.

   (a) Show that $x_1 = -x_2$.

   (b) Find x_1 and x_2.

5. For $x > 0$, show that $e^x < (1+x)^{1+x}$.

6. Given the linear fractional transformation of x into $f_1(x) = (2x-1)/(x+1)$, define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \ldots$. It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = (Ax+B)/(Cx+D)$.

7. Let S be the set of all ordered pairs of integers (m,n) satisfying $m > 0$ and $n < 0$. Let $\langle$ be a partial ordering on S defined by the statement: $(m,n) \langle (m',n')$ if and only if $m \leq m'$ and $n \leq n'$. An example is $(5,-10) \langle (8,-2)$. Now let O be a completely ordered subset of S, i.e. if $(a,b) \in O$ and $(c,d) \in O$, then $(a,b) \langle (c,d)$ or $(c,d) \langle (a,b)$. Also let $\mathcal{O}$ denote the collection of all such completely ordered sets.

   (a) Determine whether an arbitrary $O \in \mathcal{O}$ is finite.

   (b) Determine whether the cardinality $\|O\|$ of O is bounded for $O \in \mathcal{O}$.

   (c) Determine whether $\|O\|$ can be countably infinite for any $O \in \mathcal{O}$.

8. Let $z = x + iy$ be a complex number with x And y rational and with $|z| = 1$.

   (a) Find two such complex numbers.

   (b) Show that $|z^{2n} - 1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.

   (c) Show that $|z^{2n} - 1|$ is rational for every n.