29th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2007

Fill out the individual registration form

1. Evaluate $\int_0^x \frac{d\theta}{2+\tan\theta}$, where $0 \le x \le \pi/2$. Use your result to show that $\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi + \ln(9/8)}{10}$.

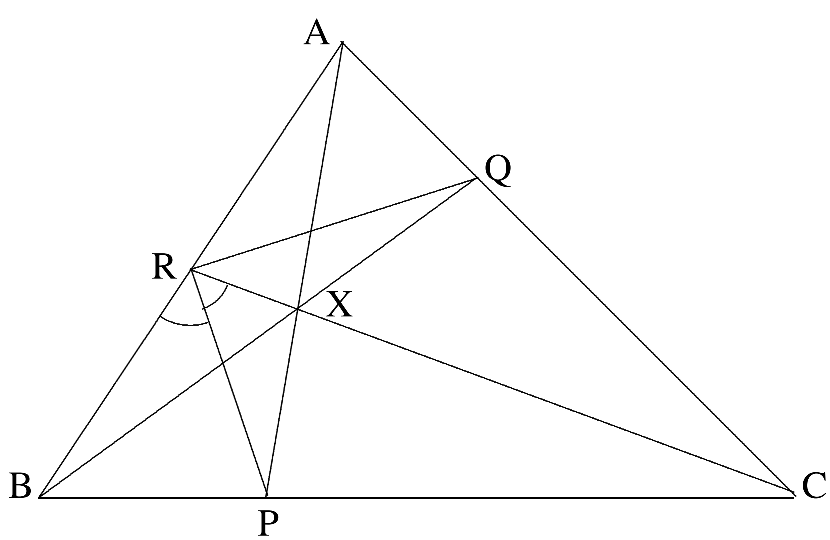
2. Given that $e^x = 1/0! + x/1! + x^2/2! + \cdots + x^n/n! + \cdots$ find, in terms of e, the exact values of

   (a) $\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \cdots + \frac{n}{(2n-1)!} + \cdots$ and

   (b) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \cdots + \frac{n}{(2n+1)!} + \cdots$

3. Solve the initial value problem $\frac{dy}{dx} = y\ln y + ye^x$, $y(0) = 1$ (i.e. find y in terms of x).

4. In the diagram below, P, Q, R are points on BC, CA, AB respectively such that the lines AP, BQ, CR are concurrent at X. Also PR bisects $\angle BRC$, i.e. $\angle BRP = \angle PRC$. Prove that $\angle PRQ = 90°$.



(Please turn over)

5. Find the third digit after the decimal point of

$$(2+\sqrt{5})^{100}\left((1+\sqrt{2})^{100}+(1+\sqrt{2})^{-100}\right).$$

For example, the third digit after the decimal point of $\pi = 3.14159\ldots$ is 1.

6. Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$ (det indicates the determinant).

7. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} n^{-(1+(\ln(\ln n))^{-2})}$ is convergent or divergent (ln denotes natural log).