24th Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., October 26, 2002

Fill out the individual registration form

1. Let $a, b$ be positive constants. Find the volume (in the first octant) which lies above the region in the $xy$-plane bounded by $x = 0, x = \pi/2, y = 0, y\sqrt{b^2 \cos^2 x + a^2 \sin^2 x} = 1$, and below the plane $z = y$.

2. Find rational numbers $a, b, c, d, e$ such that
\[
\sqrt{7 + \sqrt{40}} = a + b\sqrt{2} + c\sqrt{5} + d\sqrt{7} + e\sqrt{10}.
\]

3. Let $A$ and $B$ be nonempty subsets of $S = \{1, 2, \ldots, 99\}$ (integers from 1 to 99 inclusive). Let $a$ and $b$ denote the number of elements in $A$ and $B$ respectively, and suppose $a + b = 100$. Prove that for each integer $s$ in $S$, there are integers $x$ in $A$ and $y$ in $B$ such that $x + y = s$ or $s + 99$.

4. Let $\{1, 2, 3, 4\}$ be a set of abstract symbols on which the associative binary operation $*$ is defined by the following operation table (associative means $(a * b) * c = a * (b * c)$):

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If the operation $*$ is represented by juxtaposition, e.g., $2 * 3$ is written as $23$ etc., then it is easy to see from the table that of the four possible “words” of length two that can be formed using only 2 and 3, i.e., 22, 23, 32 and 33, exactly two, 22 and 33, are equal to 1. Find a formula for the number $A(n)$ of words of length $n$, formed by using only 2 and 3, that equal 1. From the table and the example just given for words of length two, it is clear that $A(1) = 0$ and $A(2) = 2$. Use the formula to find $A(12)$.

(Please turn over)
5. Let \( n \) be a positive integer. A bit string of length \( n \) is a sequence of \( n \) numbers consisting of 0’s and 1’s. Let \( f(n) \) denote the number of bit strings of length \( n \) in which every 0 is surrounded by 1’s. (Thus for \( n = 5 \), 11101 is allowed, but 10011 and 10110 are not allowed, and we have \( f(3) = 2 \), \( f(4) = 3 \).) Prove that \( f(n) < (1.7)^n \) for all \( n \).

6. Let \( S \) be a set of \( 2 \times 2 \) matrices with complex numbers as entries, and let \( T \) be the subset of \( S \) consisting of matrices whose eigenvalues are \( \pm 1 \) (so the eigenvalues for each matrix in \( T \) are \( \{1,1\} \) or \( \{1,-1\} \) or \( \{-1,-1\} \)). Suppose there are exactly three matrices in \( T \). Prove that there are matrices \( A, B \) in \( S \) such that \( AB \) is not a matrix in \( S \) (\( A = B \) is allowed).

7. Let \( \{a_n\}_{n \geq 1} \) be an infinite sequence with \( a_n \geq 0 \) for all \( n \). For \( n \geq 1 \), let \( b_n \) denote the geometric mean of \( a_1, \ldots, a_n \), that is \( (a_1 \ldots a_n)^{1/n} \). Suppose \( \sum_{n=1}^{\infty} a_n \) is convergent. Prove that \( \sum_{n=1}^{\infty} b_n^2 \) is also convergent.