24th Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., October 26, 2002

Fill out the individual registration form

1. Let a, b be positive constants. Find the volume (in the first octant) which lies above the region in the xy-plane bounded by $x = 0$, $x = \pi/2$, $y = 0$, $y\sqrt{b^2\cos^2 x + a^2\sin^2 x} = 1$, and below the plane $z = y$.

2. Find rational numbers a, b, c, d, e such that
$$\sqrt{7+\sqrt{40}} = a + b\sqrt{2} + c\sqrt{5} + d\sqrt{7} + e\sqrt{10}.$$

3. Let A and B be nonempty subsets of $S = \{1, 2, \ldots, 99\}$ (integers from 1 to 99 inclusive). Let a and b denote the number of elements in A and B respectively, and suppose $a + b = 100$. Prove that for each integer s in S, there are integers x in A and y in B such that $x + y = s$ or $s + 99$.

4. Let $\{1,2,3,4\}$ be a set of abstract symbols on which the associative binary operation $*$ is defined by the following operation table (associative means $(a*b)*c = a*(b*c)$):

$*$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

If the operation $*$ is represented by juxtaposition, e.g., $2*3$ is written as 23 etc., then it is easy to see from the table that of the four possible "words" of length two that can be formed using only 2 and 3, i.e., 22, 23, 32 and 33, exactly two, 22 and 33, are equal to 1. Find a formula for the number $A(n)$ of words of length n, formed by using only 2 and 3, that equal 1. From the table and the example just given for words of length two, it is clear that $A(1) = 0$ and $A(2) = 2$. Use the formula to find $A(12)$.

(Please turn over)

5. Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0's and 1's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1's. (Thus for $n = 5$, 11101 is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2$, $f(4) = 3$.) Prove that $f(n) < (1.7)^n$ for all n.

6. Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1,1\}$ or $\{1,-1\}$ or $\{-1,-1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in S ($A = B$ is allowed).

7. Let $\{a_n\}_{n \geq 1}$ be an infinite sequence with $a_n \geq 0$ for all n. For $n \geq 1$, let b_n denote the geometric mean of $a_1, \ldots, a_n$, that is $(a_1 \ldots a_n)^{1/n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.