22nd Annual

Virginia Tech Regional Mathematics Contest

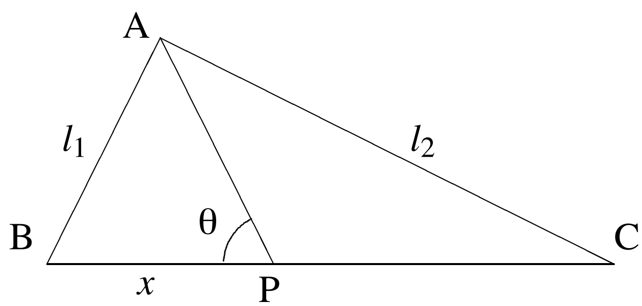
From 8:30 a.m. to 11:00 a.m., October 28, 2000

Fill out the individual registration form

1. Evaluate $\int_0^\alpha \frac{d\theta}{5-4\cos\theta}$ (your answer will involve inverse trig functions; you may assume that $0 \le \alpha < \pi$). Use your answer to show that $\int_0^{\pi/3} \frac{d\theta}{5-4\cos\theta} = \frac{2\pi}{9}$.

2. Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (i.e. the sum of the entries on the main diagonal) is an integer.

3. Consider the initial value problem $y' = y^2 - t^2$; $y(0) = 0$ (where $y' = dy/dt$). Prove that $\lim_{t\to\infty} y'(t)$ exists, and determine its value.

4. 

   In the above diagram, $l_1 = \overline{AB}$, $l_2 = \overline{AC}$, $x = \overline{BP}$, and $l = \overline{BC}$, where $\overline{AB}$ indicates the length of AB. Prove that $l_2 - l_1 = \int_0^l \cos(\theta(x))\,dx$.

5. Two diametrically opposite points P, Q lie on an infinitely long cylinder which has radius $2/\pi$. A piece of string with length 8 has its ends joined to P, is wrapped once round the outside of the cylinder, and then has its midpoint joined to Q (so there is length 4 of the string on each side of the cylinder). A paint brush is attached to the string so that it can slide along the full length the string. Find the area of the outside surface of the cylinder which can be painted by the brush.

6. Let a_n $(n \geq 1)$ be the sequence of numbers defined by the recurrence relation

$$a_1 = 1, \quad a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2a_{n-2} + a_1a_{n-1}$$

(so $a_2 = a_1^2 = 1, a_3 = 2a_1a_2 = 2$ etc.). Prove that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n a_n = \frac{1}{3}$.

7. Two types of domino-like rectangular tiles, [•• | •] and [•• | ••], are available. The first type may be rotated end-to-end to produce a tile of type [• | ••]. Let $A(n)$ be the number of distinct chains of n tiles, placed end-to-end, that may be constructed if abutting ends are required to have the same number of dots.

Example $A(2) = 5$, since the following five chains of length two, and no others, are allowed.

[• | ••][•• | •], [• | ••][•• | ••], [•• | •][• | ••],
[•• | ••][•• | •], [•• | ••][•• | ••].

(a) Find $A(3)$ and $A(4)$.

(b) Find, with proof, a three-term recurrence formula for $A(n+2)$ in terms of $A(n+1)$ and $A(n)$, for $n = 1, 2, \ldots$, and use it to find $A(10)$.