1st Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 10, 1979

Fill out the individual registration form

1. Show that the right circular cylinder of volume V which has the least surface area is the one whose diameter is equal to its altitude. (The top and bottom are part of the surface.)

2. Let S be a set which is closed under the binary operation $\circ$, with the following properties:

   (i) there is an element $e \in S$ such that $a \circ e = e \circ a = a$, for each $a \in S$,

   (ii) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$, for all $a, b, c, d \in S$.

   Prove or disprove:

   (a) $\circ$ is associative on S

   (b) $\circ$ is commutative on S

3. Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\bar{A}$ be its complex conjugate. Let $B = A\bar{A} + I$, where I is the $n \times n$ identity matrix.

   (a) Prove or disprove: $A^{-1}BA = \bar{B}$.

   (b) Prove or disprove: the determinant of $A\bar{A} + I$ is real.

4. Let $f(x)$ be continuously differentiable on $(0, \infty)$ and suppose $\lim_{x \to \infty} f'(x) = 0$. Prove that $\lim_{x \to \infty} f(x)/x = 0$.

5. Show, for all positive integers $n = 1, 2, \ldots$, that 14 divides $3^{4n+2} + 5^{2n+1}$.

6. Suppose $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Determine whether $\sum_{n=1}^{\infty} a_n/S_n^2$ converges, where $S_n = a_1 + a_2 + \cdots + a_n$.

7. Let S be a finite set of non-negative integers such that $|x - y| \in S$ whenever $x, y \in S$.

   (a) Give an example of such a set which contains ten elements.

(b) If A is a subset of S containing more than two-thirds of the elements of S, prove or disprove that *every* element of S is the sum or difference of two elements from A.

8. Let S be a finite set of polynomials in two variables, x and y. For n a positive integer, define $\Omega_n(S)$ to be the collection of all expressions $p_1 p_2 \dots p_k$, where $p_i \in S$ and $1 \leq k \leq n$. Let $d_n(S)$ indicate the maximum number of linearly independent polynomials in $\Omega_n(S)$. For example, $\Omega_2(\{x^2, y\}) = \{x^2, y, x^2y, x^4, y^2\}$ and $d_2(\{x^2, y\}) = 5$.

   (a) Find $d_2(\{1, x, x+1, y\})$.

   (b) Find a closed formula in n for $d_n(\{1, x, y\})$.

   (c) Calculate the least upper bound over all such sets of $\overline{\lim}_{n\to\infty} \dfrac{\log d_n(S)}{\log n}$. ($\overline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_n, a_{n+1}, \dots\})$, where sup means supremum or least upper bound.)