19th Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., November 1, 1997

Fill out the individual registration form

1. Evaluate $\iint_D \frac{x^3}{x^2+y^2}\,dA$, where D is the half disk given by $(x-1)^2+y^2 \leq 1, y \geq 0$.

2. Suppose that $r_1 \neq r_2$ and $r_1 r_2 = 2$. If r_1 and r_2 are roots of
$$x^4 - x^3 + ax^2 - 8x - 8 = 0,$$
find r_1, r_2 and a. (Do not assume that they are real numbers.)

3. Suppose that you are in charge of taking ice cream orders for a class of 100 students. If each student orders exactly one flavor from Vanilla, Strawberry, Chocolate and Pecan, how many different combinations of flavors are possible for the 100 orders you are taking. Here are some examples of possible combinations. You do not distinguish between individual students.
   (i) $V = 30, S = 20, C = 40, P = 10$.
   (ii) $V = 80, S = 0, C = 20, P = 0$.
   (iii) $V = 0, S = 0, C = 0, P = 100$.

4. A business man works in New York and Los Angeles. If he is in New York, each day he has four options; to remain in New York, or to fly to Los Angeles by either the 8:00 a.m., 1:00 p.m. or 6:00 p.m. flight. On the other hand if he is in Los Angeles, he has only two options; to remain in Los Angeles, or to fly to New York by the 8:00 a.m. flight. In a 100 day period he has to be in New York both at the beginning of the first day of the period, and at the end of the last day of the period. How many different possible itineraries does the business man have for the 100 day period (for example if it was for a 2 day period rather than a 100 day period, the answer would be 4)?

5. The VTRC bus company serves cities in the USA. A subset S of the cities is called well-served if it has at least three cities and from every city A in S,

one can take a nonstop VTRC bus to at least two different other cities B and C in S (though there is not necessarily a nonstop VTRC bus from B to A or from C to A). Suppose there is a well-served subset S. Prove that there is a well-served subset T such that for any two cities A, B in T, one can travel by VTRC bus from A to B, stopping only at cities in T.

6. A disk of radius 1 cm. has a small hole at a point half way between the center and the circumference. The disk is lying inside a circle of radius 2 cm. A pen is put through the hole in the disk, and then the disk is moved once round the inside of the circle, keeping the disk in contact with the circle without slipping, so the pen draws a curve. What is the area enclosed by the curve?

7. Let J be the set of all sequences of real numbers, and let A, L and P be three mappings from J to J defined as follows. If $x = \{x_n\} = \{x_0, x_1, x_2, \dots\} \in J$, then

$$\begin{aligned} Ax &= \{x_n + 1\} = \{x_0 + 1,\ x_1 + 1,\ x_2 + 1, \dots\}, \\ Lx &= \{1, x_0, x_1, x_2, \dots\}, \\ Px &= \{\sum_{k=0}^{n} x_k\}. \end{aligned}$$

Finally, define the composite mapping T on J by $Tx = L \circ A \circ Px$. In the following, let $y = \{1, 1, 1, \dots\}$.

   (a) Write down $T^2 y$, giving the first eight terms of the sequence and a closed formula for the n-th term.

   (b) Assuming that $z = \{z_n\} = \lim_{i \to \infty} T^i y$ exists, conjecture the general form for z_n, and prove your conjecture.