**18th Annual**
**Virginia Tech Regional Mathematics Contest**
From 9:00 a.m. to 11:30 a.m., October 26, 1996

**Fill out the individual registration form**

1. Evaluate $\int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^4+2x^2y^2+y^4)}\, dxdy$.

2. For each rational number $r$, define $f(r)$ to be the smallest positive integer $n$ such that $r = m/n$ for some integer $m$, and denote by $P(r)$ the point in the $(x,y)$ plane with coordinates $P(r) = (r, 1/f(r))$. Find a necessary and sufficient condition that, given two rational numbers $r_1$ and $r_2$ such that $0 < r_1 < r_2 < 1$,
$$P\Big(\frac{r_1 f(r_1) + r_2 f(r_2)}{f(r_1) + f(r_2)}\Big)$$
will be the point of intersection of the line joining $(r_1, 0)$ and $P(r_2)$ with the line joining $P(r_1)$ and $(r_2, 0)$.

3. Solve the differential equation $y^y = e^{dy/dx}$ with the initial condition $y = e$ when $x = 1$.

4. Let $f(x)$ be a twice continuously differentiable in the interval $(0,\infty)$. If
$$\lim_{x\to\infty}(x^2 f''(x) + 4xf'(x) + 2f(x)) = 1,$$
find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} xf'(x)$. Do **not** assume any special form of $f(x)$. Hint: use l'Hôpital's rule.

5. Let $a_i$, $i = 1,2,3,4$, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers $b_i$, $i = 1,2,3$, the equation
$$a_1 + b_1 x + 3a_2 x^2 + b_2 x^3 + 5a_3 x^4 + b_3 x^5 + 7a_4 x^6 = 0$$
has at least one real root which is on the interval $-1 \le x \le 1$.

6. There are $2n$ balls in the plane such that no three balls are on the same line and such that no two balls touch each other. $n$ balls are red and the other $n$ balls are green. Show that there is at least one way to draw $n$ line segments by connecting each ball to a unique different colored ball so that no two line segments intersect.

7. Let us define

$$f_{n,0}(x) = x + \frac{\sqrt{x}}{n} \qquad \text{for } x > 0,\ n \geq 1,$$
$$f_{n,j+1}(x) = f_{n,0}(f_{n,j}(x)), \qquad j = 0, 1, \ldots, n-1.$$

Find $\lim_{n\to\infty} f_{n,n}(x)$ for $x > 0$.