18th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 26, 1996

Fill out the individual registration form

1. Evaluate $\int_{0}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{x^2+y^2}} xe^{(x^4+2x^2y^2+y^4)} \, dxdy$.

2. For each rational number $r$, define $f(r)$ to be the smallest positive integer $n$ such that $r = m/n$ for some integer $m$, and denote by $P(r) = (r, 1/f(r))$. Find a necessary and sufficient condition that, given two rational numbers $r_1$ and $r_2$ such that $0 < r_1 < r_2 < 1$, $P \left( \frac{r_1f(r_1) + r_2f(r_2)}{f(r_1) + f(r_2)} \right)$ will be the point of intersection of the line joining $(r_1, 0)$ and $P(r_2)$ with the line joining $P(r_1)$ and $(r_2, 0)$.

3. Solve the differential equation $y^y = e^{dy/dx}$ with the initial condition $y = e$ when $x = 1$.

4. Let $f(x)$ be a twice continuously differentiable in the interval $(0, \infty)$. If
   $$\lim_{x \to \infty} (x^2f''(x) + 4xf'(x) + 2f(x)) = 1,$$
   find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} xf'(x)$. Do not assume any special form of $f(x)$. Hint: use l'Hôpital’s rule.

5. Let $a_i, i = 1, 2, 3, 4$, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers $b_i, i = 1, 2, 3$, the equation
   $$a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6 = 0$$
   has at least one real root which is on the interval $-1 \leq x \leq 1$.

6. There are $2n$ balls in the plane such that no three balls are on the same line and such that no two balls touch each other. $n$ balls are red and the other $n$ balls are green. Show that there is at least one way to draw $n$ line segments by connecting each ball to a unique different colored ball so that no two line segments intersect.
7. Let us define

\[ f_{n,0}(x) = x + \frac{\sqrt{x}}{n} \quad \text{for } x > 0, \quad n \geq 1, \]

\[ f_{n,j+1}(x) = f_{n,0}(f_{n,j}(x)), \quad j = 0, 1, \ldots, n-1. \]

Find \( \lim_{n \to \infty} f_{n,n}(x) \) for \( x > 0 \).