17th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 28, 1995

Fill out the individual registration form

1. Evaluate $\int_0^3 \int_0^2 \frac{1}{1+(\max(3x,2y))^2\,dxdy}$.

2. Let $\mathbb{R}^2$ denote the xy-plane, and define $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\theta(x,y) = (4x-3y+1, 2x-y+1)$. Determine $\theta^{100}(1,0)$, where θ^{100} indicates applying θ 100 times.

3. Let $n \geq 2$ be a positive integer and let $f(x)$ be the polynomial

   $$1-(x+x^2+\cdots+x^n)+(x+x^2+\cdots+x^n)^2-\cdots+(-1)^n(x+x^2+\cdots+x^n)^n.$$

   If r is an integer such that $2 \leq r \leq n$, show that the coefficient of x^r in $f(x)$ is zero.

4. Let $\tau = (1+\sqrt{5})/2$. Show that $[\tau^2 n] = [\tau[\tau n]+1]$ for every positive integer n. Here $[r]$ denotes the largest integer that is not larger than r.

5. Let $\mathbb{R}$ denote the real numbers, and let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a map with the property that $x > y$ implies $(\theta(x))^3 > \theta(y)$. Prove that $\theta(x) > -1$ for all x, and that $0 \leq \theta(x) \leq 1$ for at most one value of x.

6. A straight rod of length 4 inches has ends which are allowed to slide along the perimeter of a square whose sides each have length 12 inches. A paint brush is attached to the rod so that it can slide between the two ends of the rod. Determine the total possible area of the square which can be painted by the brush.

7. If n is a positive integer larger than 1, let $n = \prod p_i^{k_i}$ be the unique prime factorization of n, where the p_i's are distinct primes, $2, 3, 5, 7, 11, \ldots$, and define $f(n)$ by $f(n) = \sum k_i p_i$ and $g(n)$ by $g(n) = \lim_{m\to\infty} f^m(n)$, where f^m is meant the m-fold application of f. Then n is said to have *property H* if $n/2 < g(n) < n$.

   (i) Evaluate $g(100)$ and $g(10^{10})$.

   (ii) Find all positive odd integers larger than 1 that have property H.