16th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 29, 1994

Fill out the individual registration form

1. Evaluate \( \int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2} \, dz \, dy \, dx \).

2. Let \( f \) be continuous real function, strictly increasing in an interval \([0,a]\) such that \( f(0) = 0 \). Let \( g \) be the inverse of \( f \), i.e., \( g(f(x)) = x \) for all \( x \) in \([0,a]\). Show that for \( 0 \leq x \leq a, 0 \leq y \leq f(a) \), we have
\[
xy \leq \int_0^x f(t) \, dt + \int_0^y g(t) \, dt.
\]

3. Find all continuously differentiable solutions \( f(x) \) for
\[
f(x)^2 = \int_0^x \left( f(t)^2 - f(t)^4 + (f'(t))^2 \right) \, dt + 100
\]
where \( f(0)^2 = 100 \).

4. Consider the polynomial equation \( ax^4 + bx^3 + x^2 + bx + a = 0 \), where \( a \) and \( b \) are real numbers, and \( a > 1/2 \). Find the maximum possible value of \( a + b \) for which there is at least one positive real root of the above equation.

5. Let \( f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R} \) be a function which satisfies \( f(0,0) = 1 \) and
\[
f(m,n) + f(m+1,n) + f(m,n+1) + f(m+1,n+1) = 0
\]
for all \( m,n \in \mathbb{Z} \) (where \( \mathbb{Z} \) and \( \mathbb{R} \) denote the set of all integers and all real numbers, respectively). Prove that \( |f(m,n)| \geq 1/3 \), for infinitely many pairs of integers \((m,n)\).

6. Let \( A \) be an \( n \times n \) matrix and let \( \alpha \) be an \( n \)-dimensional vector such that \( A\alpha = \alpha \). Suppose that all the entries of \( A \) and \( \alpha \) are positive real numbers. Prove that \( \alpha \) is the only linearly independent eigenvector of \( A \) corresponding to the eigenvalue 1. Hint: if \( \beta \) is another eigenvector, consider the minimum of \( \alpha_i/|\beta_i|, i = 1, \ldots, n \), where the \( \alpha_i \)'s and \( \beta_i \)'s are the components of \( \alpha \) and \( \beta \), respectively.
7. Define $f(1) = 1$ and $f(n + 1) = 2\sqrt{f(n)^2 + n}$ for $n \geq 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^{N} f(n)^2$.

8. Let a sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers be defined by $x_0 = 10, x_1 = 29$ and $x_{n+2} = \frac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n}$.