16th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 29, 1994

Fill out the individual registration form

1. Evaluate $\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2}\, dz dy dx$.

2. Let f be continuous real function, strictly increasing in an interval $[0,a]$ such that $f(0) = 0$. Let g be the inverse of f, i.e., $g(f(x)) = x$ for all x in $[0,a]$. Show that for $0 \le x \le a, 0 \le y \le f(a)$, we have
$$xy \le \int_0^x f(t)\, dt + \int_0^y g(t)\, dt.$$

3. Find all continuously differentiable solutions $f(x)$ for
$$f(x)^2 = \int_0^x \left(f(t)^2 - f(t)^4 + (f'(t))^2\right) dt + 100$$
where $f(0)^2 = 100$.

4. Consider the polynomial equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a and b are real numbers, and $a > 1/2$. Find the maximum possible value of $a+b$ for which there is at least one positive real root of the above equation.

5. Let $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0,0) = 1$ and
$$f(m,n) + f(m+1,n) + f(m,n+1) + f(m+1,n+1) = 0$$
for all $m,n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m,n)| \ge 1/3$, for infinitely many pairs of integers (m,n).

6. Let A be an $n \times n$ matrix and let α be an n-dimensional vector such that $A\alpha = \alpha$. Suppose that all the entries of A and α are positive real numbers. Prove that α is the only linearly independent eigenvector of A corresponding to the eigenvalue 1. Hint: if β is another eigenvector, consider the minimum of $\alpha_i/|\beta_i|$, $i = 1,\ldots,n$, where the α_i's and β_i's are the components of α and β, respectively.

7. Define $f(1) = 1$ and $f(n+1) = 2\sqrt{f(n)^2 + n}$ for $n \geq 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^{N} f(n)^2$.

8. Let a sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers be defined by $x_0 = 10, x_1 = 29$ and $x_{n+2} = \dfrac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\displaystyle\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n}$.