14th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 31, 1992

Fill out the individual registration form

1. Find the inflection point of the graph of $F(x) = \int_0^{x^3} e^{t^2}\,dt$, for $x \in \mathbb{R}$.

2. Assume that $x_1 > y_1 > 0$ and $y_2 > x_2 > 0$. Find a formula for the shortest length l of a planar path that goes from (x_1, y_1) to (x_2, y_2) and that touches both the x-axis and the y-axis. Justify your answer.

3. Let $f_n(x)$ be defined recursively by

$$f_0(x) = x, \quad f_1(x) = f(x), \quad f_{n+1}(x) = f(f_n(x)), \quad \text{for} \quad n \geq 0,$$

where $f(x) = 1 + \sin(x-1)$.

   (i) Show that there is a unique point x_0 such that $f_2(x_0) = x_0$.

   (ii) Find $\sum_{n=0}^{\infty} \frac{f_n(x_0)}{3^n}$ with the above x_0.

4. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1}^2 = 1 + t_n$, for $n \geq 1$. Show that t_n is increasing in n and find $\lim_{n\to\infty} t_n$.

5. Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{100}. You have to find all four entries.

6. Let $p(x)$ be the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that if $p(r) = 0$, then

$$\frac{p(x)}{x-r} - 2\frac{p(x+1)}{x+1-r} + \frac{p(x+2)}{x+2-r} = 2$$

for all x except $x = r$, $r-1$ and $r-2$.

7. Find $\lim_{n\to\infty} \frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2 \log n}$.

8. Some goblins, N in number, are standing in a row while "trick-or-treat"ing. Each goblin is at all times either $2'$ tall or $3'$ tall, but can change spontaneously from one of these two heights to the other at will. While lined up in such a row, a goblin is called a **Local Giant Goblin** (LGG) if he/she/it is not standing beside a taller goblin. Let $G(N)$ be the total of all occurrences of LGG's as the row of N goblins transmogrifies through all possible distinct configurations, where height is the only distinguishing characteristic. As an example, with $N = 2$, the distinct configurations are $\hat{2}\hat{2}$, $2\hat{3}$, $\hat{3}2$, $\hat{3}\hat{3}$, where a cap indicates an LGG. Thus $G(2) = 6$.

   (i) Find $G(3)$ and $G(4)$.

   (ii) Find, with proof, the general formula for $G(N)$, $N = 1, 2, 3, \ldots$.