12th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 20, 1990

Fill out the individual registration form

1. Three pasture fields have areas of 10/3, 10 and 24 acres, respectively. The fields initially are covered with grass of the same thickness and new grass grows on each at the same rate per acre. If 12 cows eat the first field bare in 4 weeks and 21 cows eat the second field bare in 9 weeks, how many cows will eat the third field bare in 18 weeks? Assume that all cows eat at the same rate. (From *Math Can be Fun* by Ya Perelman.)

2. A person is engaged in working a jigsaw puzzle that contains 1000 pieces. It is found that it takes 3 minutes to put the first two pieces together and that when x pieces have been connected it takes $\dfrac{3(1000-x)}{1000+x}$ minutes to connect the next piece. Determine an accurate estimate of the time it takes to complete the puzzle. Give both a formula and an approximate numerical value in hours. (You may find useful the approximate value $\ln 2 = .69$.)

3. Let f be defined on the natural numbers as follows: $f(1) = 1$ and for $n > 1$, $f(n) = f(f(n-1)) + f(n - f(n-1))$. Find, with proof, a simple explicit expression for $f(n)$ which is valid for all $n = 1, 2, \ldots$.

4. Suppose that $P(x)$ is a polynomial of degree 3 with integer coefficients and that $P(1) = 0$, $P(2) = 0$. Prove that at least one of its four coefficients is equal to or less than -2.

5. Determine all real values of p for which the following series converge.

$$\text{(a)} \sum_{n=1}^{\infty} (\sin \frac{1}{n})^p \quad \text{(b)} \sum_{n=1}^{\infty} |\sin n|^p$$

6. The number of individuals in a certain population (in arbitrary real units) obeys, at discrete time intervals, the equation

$$y_{n+1} = y_n(2 - y_n) \quad \text{for } n = 0, 1, 2, \ldots,$$

where y_0 is the initial population.

(a) Find all "steady-state" solutions y^* such that, if $y_0 = y^*$, then $y_n = y^*$ for $n = 1, 2, \ldots$.

(b) Prove that if y_0 is any number in $(0, 1)$, then the sequence $\{y_n\}$ converges monotonically to one of the steady-state solutions found in (a).

7. Let the following conditions be satisfied:

(i) $f = f(x)$ and $g = g(x)$ are continuous functions on $[0, 1]$,

(ii) there exists a number a such that $0 < f(x) \leq a < 1$ on $[0, 1]$,

(iii) there exists a number u such that $\max_{0 \leq x \leq 1} (g(x) + uf(x)) = u$.

Find constants A and B such that $F(x) = \dfrac{Ag(x)}{f(x) + B}$ is a continuous function on $[0, 1]$ satisfying $\max_{0 \leq x \leq 1} F(x) = u$, and prove that your function has the required properties.

8. Ten points in space, no three of which are collinear, are connected, each one to all the others, by a total of 45 line segments. The resulting framework F will be "disconnected" into two disjoint nonempty parts by the removal of one point from the interior of each of the 9 segments emanating from any one vertex of f. Prove that F cannot be similarly disconnected by the removal of only 8 points from the interiors of the 45 segments.