A RECTANGULAR IMMERSED FINITE ELEMENT SPACE FOR INTERFACE PROBLEMS*

TAO LIN†, YANPING LIN‡, ROBERT ROGERS§, AND M. LYNNE RYAN¶

Abstract. We consider an immersed finite element space for boundary value problems of partial differential equations with discontinuous coefficients. The basis functions in this space are constructed as piecewise bi-linear polynomials that satisfy jump conditions approximately (or even exactly in certain situations). The mesh in this space does not have to be aligned with the interface because the interface is allowed to pass through its elements. Therefore a structured Cartesian mesh can be used to solve boundary value problems with arbitrary interfaces. Numerical results are presented to show the convergence of the Galerkin method based on this space.

Key words. interface problems, immersed interface, finite element, convergence rates.

AMS subject classifications. 65N15, 65N30, 65N50, 35R05

1. Introduction. In this paper we consider an immerse finite element method based on a rectangular mesh for the following boundary value problem:

$$-\nabla \cdot \big(\beta \nabla u\big) = f, \quad (x,y) \in \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = g, \tag{1.2}$$

together with the jump conditions on the interface Γ:

$$[u]\,|_\Gamma = 0, \tag{1.3}$$

$$\left[\beta \frac{\partial u}{\partial n}\right] |_\Gamma = 0. \tag{1.4}$$

Here, we assume that $\Omega \subset R^2$ is a rectangular domain (or a union of several rectangular domains), the interface Γ is a smooth curve separating Ω into two subdomains Ω^-, Ω^+ such that $\Omega = \Omega^- \cup \Omega^+ \cup \Gamma$, see Figure 1.

The coefficient $\beta(x,y)$ is a piecewise constant function defined by

$$\beta(x,y) = \begin{cases} \beta^-, & (x,y) \in \Omega^-, \\ \beta^+, & (x,y) \in \Omega^+. \end{cases}$$

Many applications and numerical methods involve solving such an interface problem, for example, the projection method for solving two phase flow problem [11, 19], Navier-Stokes equations [1, 2, 5], and Hele-Shaw flow [7, 8], to name just a few.

Standard numerical methods such as the Galerkin finite element method based on linear polynomials can be used to solve such an interface problem, see [3, 4] and the references therein. However, to maintain the best possible convergence rate, the mesh used in the standard Galerkin finite element method has to be formed in a way such that the interface is allowed to intersect with edges of an element only through its

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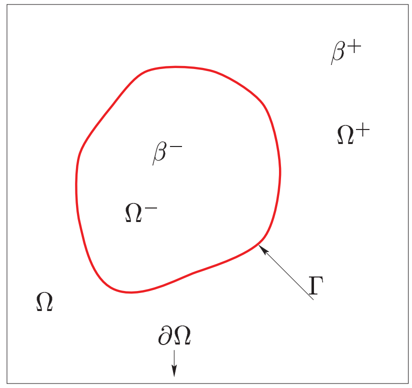

FIG. 1.1. *Geometry of the BVP.*

vertices. This restriction will prevent the Galerkin method from working efficiently for those applications with a changing interface, because the mesh needs to be reformed over and over again.

Immersed finite elements have been introduced recently to alleviate the above limitation, see [6, 7, 9, 10, 12, 13, 16, 17, 18, 20]. A key feature of immersed finite elements is that elements in a mesh do not have to be aligned with the interface so that meshes with simple structure might be used for solving a boundary value problem with an arbitrary interface. Here elements in a mesh are naturally separated into two classes: non-interface elements and interface elements. Interface elements are those through whose interior the interface passes; the non-interface elements are those otherwise. While standard finite element functions can be used in non-interface elements, macro local basis functions are constructed in every interface element such that the interface jump conditions can be satisfied approximately/exactly. Immersed finite elements based on a triangular mesh have been discussed in [15, 6, 14]. Our intention here is to investigate the immerse finite elements defined by a rectangular mesh.

This paper is organized as follows. In Section 2, we will introduce an immersed finite element space based on a rectangular mesh. Basic features of this finite element space will be presented in this section. Section 3 reports some numerical experiments we have carried out with this immersed finite element space.

2. A rectangular immersed finite element space. Withour loss of generality, we assume that the edges of Ω are parallel to the $x-y$ axes. A mesh $\mathcal{T}_h = \{T_h\}$ is then formed by lines also parallel to $x-y$ axes so that each element T_h in $\mathcal{T}_h$ is a square with edges of length h.

We first consider local basis functions in a typical element $T_h \in \mathcal{T}_h$. Assume that the four vertices of T_h are $A_i, i = 1, 2, 3, 4$, with $A_i = (x_i, y_i)^t$. In a non-interface element T_h, we just let $S_h(T_h)$ be the standard local finite element space formed by bi-linear polynomials:

$$S_h(T_h) = span\left\{\phi_i \mid \phi_i \text{ is bilinear and } \phi_i(A_j) = \left\{\begin{array}{ll} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{array}\right. \quad i, j = 1, 2, 3, 4.\right\}$$

For an interface element T_h, we assume that the interface meets its edges at the points D and E, see Figure 2.1. There are two types of interface elements. Type I are those for which the interface intersects two of its adjacent edges; Type II are those for which the interface intersects two of its opposite edgeds.

Note that the interface Γ separates T_h into T_h^+ and T_h^-, and we can use this natural partition of T_h to introduce four piecewisely defined local nodal basis functions ϕ_i as follows:

$$\phi_i(x,y) = \begin{cases} \phi_i^+(x,y), & \text{if } (x,y) \in T_h^+, \\ \phi_i^-(x,y), & \text{if } (x,y) \in T_h^-, \end{cases} \quad i = 1,2,3,4,$$

with bilinear functions $\phi_i^s, s = +,-$ determined by the following conditions:

(B_1) Nodal values: for $i, j = 1,2,3,4$,

$$\phi_i(A_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

(B_2) The continuity on the line $\overline{DE}$:

$$\phi_i^+(P_j) = \phi_i^-(P_j), \quad i = 1,2,3,4, \quad j = 1,2,3,$$

where $P_1 = D, P_2 = E, P_3 = (D+E)/2$.

(B_3) Flux continuity on the line $\overline{DE}$:

$$\int_{\overline{DE}} \left(\beta^+ \frac{\partial \phi_i^+}{\partial \mathbf{n}_{\overline{DE}}} - \beta^- \frac{\partial \phi_i^-}{\partial \mathbf{n}_{\overline{DE}}} \right) ds = 0,$$

where $\dfrac{\partial \phi_i^s}{\partial \mathbf{n}_{\overline{DE}}}, s = +,-$ is the normal derivative of ϕ_i^s along the line segment $\overline{DE}$.

Note that ϕ_i^s has 4 coefficients, and it can be shown that the 8 equations above are enough to determine a nodal basis function ϕ_i. Using these local finite element spaces, we can define a global basis function $\psi_N(x,y)$ piecewisely for each node $p_N = (x_N, y_N)^t$ in the mesh $\mathcal{T}_h$ such that

1. $\psi_N|_{T_h} \in S_h(T_h)$ for any $T_h \in \mathcal{T}_h$.
2. $$\psi_N(p_M) = \begin{cases} 1, & \text{if } N = M, \\ 0, & \text{if } N \neq M, \end{cases}$$

   where p_M is a node of $\mathcal{T}_h$.

Figure 2.2 presents a plot of a typical global nodal basis function involving interface elements.

Finally, the immersed finite element space in Ω is defined by

$$S_h(\Omega) = span\{\psi_N \mid p_N \text{ is a node of } \mathcal{T}_h\}.$$

We observe that this space has the following features:

- For a mesh $\mathcal{T}_h$, the finite element space $S_h(\Omega)$ has the same number of nodal basis functions as that formed by the usual bi-linear polynomials.
- For a mesh $\mathcal{T}_h$ fine enough, most of its elements are non-interface elements, and most of the nodal basis functions of $S_h(\Omega)$ are just the usual bi-linear nodal basis functions except for few nodes in the vicinity of the interface Γ.
- For any $\phi \in S_h(\Omega)$, we have

  $$\phi|_{\Omega \setminus \Omega'} \in H^1(\Omega \setminus \Omega'), \tag{2.1}$$

  where Ω' is the union of interface elements.

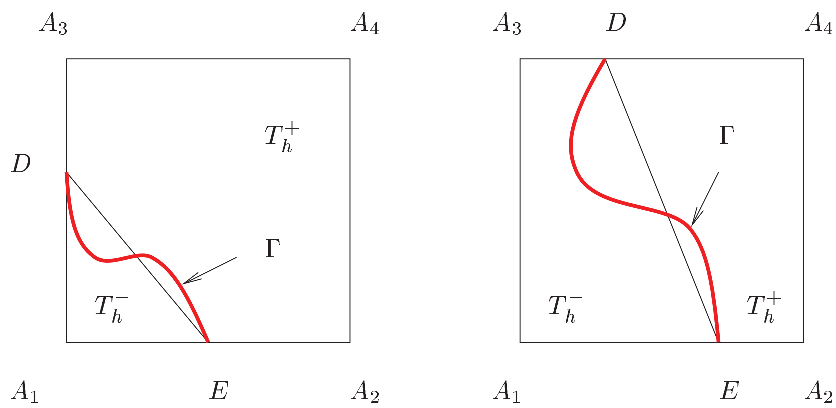

FIG. 2.1. *Two typical interface elements. The element on the left is of Type I while the one on the right is of Type II.*

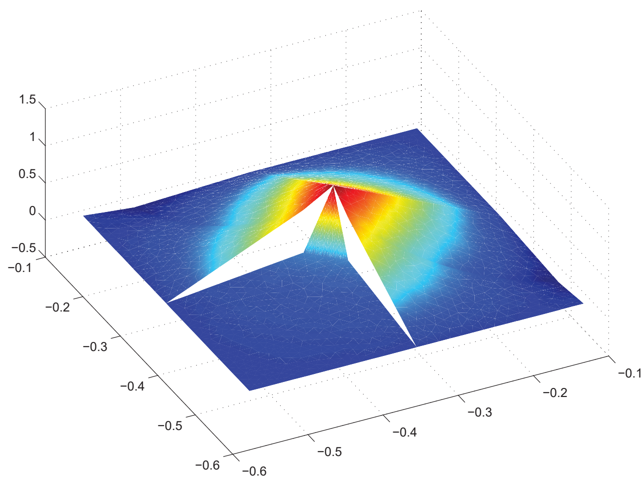

FIG. 2.2. *A typical global nodal basis function involving interface elements.*

We now give a group of propositions describing basic properties of the immersed finite element space. Because of the page limitation, we will give no proof or just an outline of proof for these results; the details will be presented in a forthcoming paper.

LEMMA 2.1. *Assume that T_h is a Type I interface element. Then any function $\phi \in S_h(T_h)$ is uniquely determined by its values at the vertices of T_h.*

Proof. Without loss of generality, we assume that T_h is the reference element whose vertices are

$$A_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the interface passes the edges of T_h at

$$D = \begin{pmatrix} 0 \\ \hat{b} \end{pmatrix}, E = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix}.$$

Let

$$\phi_i(x,y) = \begin{cases} a_i^- + b_i^- x + c_i^- y + d_i^- xy, & \text{if } (x,y) \in T_h^-, \\ a_i^+ + b_i^+ x + c_i^+ y + d_i^+ xy, & \text{if } (x,y) \in T_h^+. \end{cases}$$

Applying conditions (B_1)-(B_3) to this $\phi_i(x,y)$ leads to

$$A \begin{pmatrix} a_i^- \\ b_i^- \\ c_i^- \\ d_i^- \\ a_i^+ \\ b_i^+ \\ c_i^+ \\ d_i^+ \end{pmatrix} = \mathbf{b}_i, A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & R\hat{b} & R\hat{a} & R\frac{\hat{a}^2+\hat{b}^2}{2} & 0 & -\hat{b} & -\hat{a} & -\frac{\hat{a}^2+\hat{b}^2}{2} \\ 1 & 0 & \hat{b} & 0 & -1 & 0 & -\hat{b} & 0 \\ 1 & \hat{a} & 0 & 0 & -1 & -\hat{a} & 0 & 0 \\ 1 & \frac{\hat{a}}{2} & \frac{\hat{b}}{2} & \frac{\hat{a}\hat{b}}{4} & -1 & -\frac{\hat{a}}{2} & -\frac{\hat{b}}{2} & -\frac{\hat{a}\hat{b}}{4} \end{pmatrix},$$

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the result of this lemma follows from the fact that the determinant of A is nonzero. □

LEMMA 2.2. *Assume that T_h is a Type II interface element. Then any piecewise linear function $\phi \in S_h(T_h)$ is uniquely determined by its values at the vertices of T_h.*

LEMMA 2.3. *If the coefficient β has no jump, then $S_h(T_h)$ becomes the space of the standard bi-linear polynomials.*

To describe the approximation capability of the immersed finite element space $S_h(T_h)$ for an interface element T_h, we consider a space $J(T_h)$ consisting of functions $u(x,y)$ such that

$$\begin{cases} u^s \in H^2(T_h^s), \quad s = -,+, \\ u^-(P_j) = u^+(P_j), \quad j = 1,2,3, \\ \int_{\Gamma\cap T_h} (\beta^- \nabla u^- - \beta^+ \nabla u^+) \cdot \mathbf{n} ds = 0, \end{cases} \tag{2.2}$$

where

$$u(x,y) = \begin{cases} u^-(x,y), & \text{if } x \in T_h^-, \\ u^+(x,y), & \text{if } x \in T_h^+, \end{cases} \tag{2.3}$$

and $\mathbf{n}$ is the unit normal vector of $\Gamma \cap T_h$. For any $u \in J(T_h)$ we let

$$\|u\|_{2,T_h} = \sqrt{\|u\|^2_{2,T_h^-} + \|u\|^2_{2,T_h^+}},$$

$$|||u|||_{2,T_h} = |u|_{2,T_h} + \sum_{i=1}^{4} |u(x_i,y_i)|,$$

$$|u|_{2,T_h} = \sqrt{|u|^2_{2,T_h^-} + |u|^2_{2,T_h^+}}$$

Using Lemma 2.1, 2.2 and Green's formula we can obtain the following result:

LEMMA 2.4. *$|||\cdot|||_{2,T_h}$ is a norm in the space $J(T_h)$, and this norm is equivalent to $\|\cdot\|_{2,T_h}$.*

We can also show that $S_h(T_h)$ is a subspace of $J(T_h)$, and this important feature implies that every function in $S_h(T_h)$ satisfies the flux jump condition (1.4) locally in a weak sense. Furthermore, if $\Gamma \cap T_h$ is a line segment, then every function in $S_h(T_h)$ can also satisfy the function jump condition (1.3) exactly.

For any $u \in J(T_h)$, we let $Iu \in S_h(T_h)$ be such that

$$Iu(A_i) = u(A_i), \quad i = 1, 2, 3, 4,$$

and we call Iu the interpolant of u in $S_h(T_h)$. Using the standard scaling argument and Lemma 2.4, we can derive an error estimate for the interpolant given in the following theorem.

THEOREM 2.5. *For any $u \in J(T_h)$ we have*

$$\|u - Iu\|_{m,T_h} \leq Ch^{2-m} |u|_{2,T_h}, \quad 0 \leq m < 2, \tag{2.4}$$

where h is the length of the edges of T_h.

Now we consider a function u satisfying

$$u \in C(\Omega), \quad u|_{\Omega^s} \in H^2(\Omega^s), \ s = -, + \tag{2.5}$$

and

$$(\beta^- \nabla u^- - \beta^+ \nabla u^+) \cdot \mathbf{n} = 0, \tag{2.6}$$

on Γ. We define its interpolant $I_h u$ in the immersed finite element space $S_h(\Omega)$ by

$$I_h u(x, y) = u(x, y), \text{ if } (x, y) \text{ is a node of } \mathcal{T}_h.$$

From Theorem 2.5 we can easily obtain the following error estimate for $I_h u$.

THEOREM 2.6. *Assume that u satisfies the conditions (2.5) and (2.6), then*

$$\|u - I_h u\|_{0,\Omega} + h \|u - I_h u\|_{1,\Omega,h} \leq Ch^2 \|u\|_{2,\Omega}, \tag{2.7}$$

where

$$\|u\|_{m,\Omega,h} = \sum_{T_h \in \mathcal{T}_h} \|u\|_{m,T_h}.$$

3. A numerical example. We tested the rectangular immersed finite space in the Galerkin finite element method for the following boundary value problem:

$$\begin{aligned} &-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega \\ &[u]|_\Gamma = 0, \quad [\beta \frac{\partial u}{\partial n}]|_\Gamma = 0, \\ &u|_{\partial\Omega} = g, \end{aligned}$$

with

$$\beta(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Omega^-, \\ 10, & \text{if } (x, y) \in \Omega^+, \end{cases}$$

where $\Omega = [-1,1] \times [-1,1]$, Ω^- is the circle centered at $(0,0)$ with radius $r_0 = 0.5$. The function f and g are chosen so that

$$u(x,y) = \begin{cases} \dfrac{r^\alpha}{\beta^-}, & \text{if } r \le r_0, \\ \dfrac{r^\alpha}{\beta^-} + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right) r_0^\alpha, & \text{otherwise,} \end{cases} \quad \alpha = 3,\ r = \sqrt{x^2 + y^2},$$

is the exact solution. A typical mesh in the numerical experiment is in given in Figure 3. Actual errors of the related immersed finite element solutions for various step size h are given in Table 3 in which u_h is the immersed finite element solution. The data in this table indicate that

$$\|u - u_h\|_{L^2(\Omega)} \approx 0.2251\, h^{1.9830},$$
$$\|u - u_h\|_{H^1(\Omega)} \approx 0.7113\, h^{0.9923},$$

which are within our expectation.

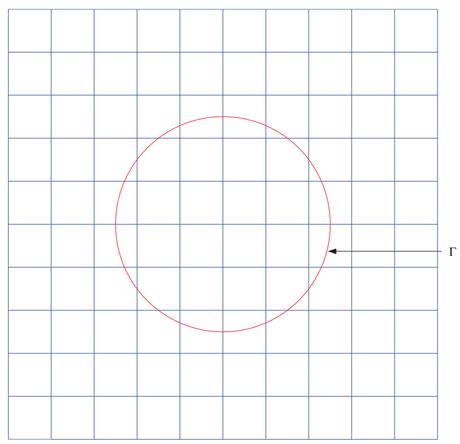

FIG. 3.1. *A typical mesh used in numerical experiments.*

h (mesh size)	$\|\|u - u_h\|\|_\infty$	$\|\|u - u_h\|\|_{L^2(\Omega)}$	$\|\|u - u_h\|\|_{H^1(\Omega)}$
1/5	5.5196×10^{-3}	8.9137×10^{-3}	1.4843×10^{-1}
1/15	2.4864×10^{-3}	1.0736×10^{-3}	4.6978×10^{-2}
1/25	1.4891×10^{-3}	3.9668×10^{-4}	2.9034×10^{-2}
1/35	1.0760×10^{-3}	2.0403×10^{-4}	1.9965×10^{-2}
1/45	8.6393×10^{-4}	1.1793×10^{-4}	1.6116×10^{-2}
1/55	7.1745×10^{-4}	7.7140×10^{-5}	1.3569×10^{-2}
1/65	6.1188×10^{-4}	5.5193×10^{-5}	1.1787×10^{-2}

TABLE 3.1
Actual errors of immersed finite element solutions.

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