PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #7 on 7/12/2018

P1) If $f, g \in \mathcal{S}(\mathbb{R})$ their convolution is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

Show the following properties for convolution.

(i) $f * g \in \mathcal{S}(\mathbb{R})$

(ii) $f * g = g * f$

(iii) $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

P2) Define for $\delta > 0$

$$K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$$

Show that $\{K_\delta\}_{\delta>0}$ is a family of good kernels as $\delta \to 0^+$.

[Hint: Part of this problem is to correctly adapt the definition of a good kernel to the real line.]

P3) For any collection of good kernels $\{K_\delta\}_{\delta>0}$ as $\delta \to 0^+$ and $f \in \mathcal{S}(\mathbb{R})$ show that

$$(f * K_\delta)(x) \to f(x)$$

uniformly in x as $\delta \to 0^+$.

P4) Show in $\mathbb{R}^d$ that

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2}$$