PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #5 on 7/9/2018

P1) Show that $|\sin(x)| \geq \left|\frac{x}{\pi/2}\right|$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

[Hint: Think about convexity and concavity of the two different functions and their geometry.]

P2) Consider the Dirichlet kernel

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Show the following properties.

(a) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)dx = 1$ for all N

(b) $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$

[Hint: Multiply $D_N(x)$ by $\sin(x/2) = \frac{1}{2i}e^{ix/2} - \frac{1}{2i}e^{-ix/2}$ and exploit a telescoping sum.]

(c) There exists a constant $C > 0$ such that

$$|D_N(x)| \leq C \min\{N, \frac{1}{|x|}\}$$

if $x \in [-\pi, \pi]$ and $N \geq 1$.

[Hint: Use the definition for the first bound, while the second bound can be obtained by using part (b) and P1).]

P3) Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, which we recall implies the following conditions.

(i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1$ for all $n \geq 1$.

(ii) There exists $M > 0$ such that

$$\int_{-\pi}^{\pi} |K_n(x)|dx \leq M \quad \text{for all } n \geq 1$$

(iii) For every $\delta > 0$

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0 \quad \text{as } n \to \infty$$

Show that if f is a bounded and integrable function on the circle then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere then show the limit is uniform.

[Hint: Let $\epsilon > 0$ be given. By continuity of f at x choose $\delta > 0$ such that $|y| < \delta$ implies $|f(x-y) - f(x)| < \epsilon$. Your goal is to make $|(f * K_n)(x) - f(x)|$ small. First show it can be written as $|\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-y) - f(x))K_n(y)dy|$. Then break this integral up into two parts, one being an integral over $|y| < \delta$ and use the continuity of f, the other being $\delta \leq |y| \leq \pi$ where you will have to use the boundedness of f along with the properties of the good kernel.]

P4) Recall that the Fejér kernel is given by

$$F_N(x) = \frac{D_0(x) + \ldots + D_{N-1}(x)}{N}$$

where $D_k(x)$ is the Dirichlet kernel from P2). Show the following properties of the Fejér kernel.

(a) $F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$
[Hint: If $\omega = e^{ix}$ we have $NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega}$.]

(b) F_N is a good kernel.
[Hint: Note that (a) shows that $F_N(x) \geq 0$.]