PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #1 on 7/2/2018

P1) Consider points labeled 1, 2 and 3 with distance $d_{1,2}$ between points 1 and 2, distance $d_{1,3}$ between points 1 and 3 and finally distance $d_{2,3}$ between points 2 and 3. One can consider the points to be vertices of a graph where edges go between all pairs of points and the edges have weights that are the distances between points. For such a graph one can consider the weighted adjacency matrix

$$\begin{bmatrix} 0 & d_{1,2} & d_{1,3} \\ d_{1,2} & 0 & d_{2,3} \\ d_{1,3} & d_{2,3} & 0 \end{bmatrix}$$

Note that such a matrix is symmetric and always with a zero diagonal. This generalizes to any number of points.

Consider a complete graph (edges between all vertices) with 4 vertices that has part of a crescent configuration structure in the sense that one edge has weight d_1, two edges have weight d_2 and three edges have weight d_3. One example of a weighted adjacency matrix for such a graph is

$$\begin{bmatrix} 0 & d_3 & d_3 & d_3 \\ d_3 & 0 & d_2 & d_2 \\ d_3 & d_2 & 0 & d_1 \\ d_3 & d_2 & d_1 & 0 \end{bmatrix}$$

Note that this does not necessarily mean there exists a crescent configuration corresponding to this matrix since it might not be geometrically realizable. However all crescent configurations have such a matrix, there just may be such matrices that do not correspond to a crescent configuration. Thus counting all such matrices gives an upper bound on how many crescent configurations you might have.

(a) How many different such matrices exist for a graph with 4 vertices?

(b) How many different such matrices exist for a graph with 5 vertices?

(c) Can you write out a general formula for how many such matrices are for a graph with N vertices?

P2) Informally a graph with part of a crescent configuration structure as in P1 is graph isomorphic to another such graph if you can relable the points in one of the graphs so that they end up with the same weighted adjacency matrix.

Can you find a relabeling of the points of the graph that has weighted adjacency matrix

$$\begin{bmatrix} 0 & d_3 & d_3 & d_3 \\ d_3 & 0 & d_1 & d_2 \\ d_3 & d_1 & 0 & d_2 \\ d_3 & d_2 & d_2 & 0 \end{bmatrix}$$

such that with the new labeling its weighted adjacency matrix becomes

$$\begin{bmatrix} 0 & d_3 & d_3 & d_3 \\ d_3 & 0 & d_2 & d_2 \\ d_3 & d_2 & 0 & d_1 \\ d_3 & d_2 & d_1 & 0 \end{bmatrix}$$

This shows that the two graphs corresponding to the two matrices are graph isomorphic. Sketch the graph corresponding to the second matrix.

P3) **Open Problem**
Find all crescent configurations on 5 points in the plane.

Strategy
In the paper Classification of crescent configurations by Durst, Hlavacek, Huynh, Miller and Palsson (https://arxiv.org/abs/1610.07836) the authors start with all matrices with part crescent configuration structure as in P1. By only considering 1 graph from each graph isomorphism class they reduce the number of graphs they need to consider down to a much lower number.

They further prune the number of graphs they need to consider by throwing out any graphs that violate one of the following conditions (that all lead to a violation of the general position condition):

(1) The configuration contains one point at the center of a circle with four or more points on this circle.
(2) The configuration contains three (ore more) isosceles triangles sharing the same base.
(3) The configuration contains four points arranged on the vertices of an isosceles trapezoid.

This brings the number of cases that have to be studied down to 51. The authors then explore which are geometrically realizable and end up with 27 different crescent configurations. This contributes many new crescent configurations on 5 points but omits any crescent configuration that has a parallelogram.

The problem with the above algorithm is that graph isomorphism does not preserve general position. In particular an isosceles trapezoid can always be inscribed into a circle and thus violates general position, however, it is graph isomorphic to a parallelogram that generically does not violate general position. Modify the algorithm by Durst et al by first throwing out all configurations that violate (1) and (2). Then look at the remaining configurations that violate (3) and determine whether they contain a parallelogram or an isosceles trapezoid and only throw out those that have the isosceles trapezoid. Then check which of the remaining configurations are geometrically realizable.

P4) **Open Problem**
Find many (all) crescent configurations on 6 points in the plane.

Strategy
The algorithm by Durst et al mentioned in P3 runs slowly for 6 points. Either improve the coding of the algorithm so that you can reasonably run it or run it so that you only test a subset of all the possible configurations. The latter suggestion means that you do not have a hope of finding all crescent configurations but it might allow you to find many new crescent configurations since the run time will be shorter when only considering a subset of all the possible options.

P5) **Open Problem**
Find many (all) crescent configurations on 4 points in $\mathbb{R}^3$.

Strategy
On 4 points it is possible that you can run the Durst et al algorithm and only exclude configurations that violate (1) and (2) and then inspect what is left by hand.

P6) **Open Problem**
Find a 6 point crescent configuration in $\mathbb{R}^3$.

Strategy
One idea is to try to brute force it and just try to guess what it should look like. This is in the style of what was done in the paper Crescent configurations by Burt, Goldstein, Manski, Miller, Palsson and Suh (https://arxiv.org/abs/1509.07220) and there you can read about what is known about crescent configurations in higher dimensions. Alternatively you can try to adapt the Durst et al algorithm to higher dimensions. What would be an anlog of (3) in higher dimensions?