Def: $f : A \to \mathbb{C}$ α-Hölder continuous if

$$[f]_\alpha := \sup_{x \neq y \in A} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty, \quad 0<\alpha \leq 1.$$

Denote the space of all such functions by $C^\alpha(A)$.

Note: Implies

$$|f(x)-f(y)| \leq [f]_\alpha |x-y|^\alpha \text{ for all } x, y \in A$$

so implies uniform continuity.

Theorem [Ex 16 in Ch. 3 for $\alpha > \frac{1}{2}$]
Let f be an α-Hölder continuous function on the circle with $0<\alpha\leq 1$. Then its Fourier series converges uniformly to f.

Proof:

$|S_N(f)(x) - f(x)|$

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y) D_N(y)\,dy - f(x) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\,dy}_{=1} \right|$$

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-y)-f(x)) D_N(y)\,dy \right|$$

$$\leq \underbrace{\left| \frac{1}{2\pi}\int_{-\delta}^{\delta} (f(x-y)-f(x)) D_N(y)\,dy \right|}_{A} + \underbrace{\left| \frac{1}{2\pi}\int_{\delta}^{\pi} (f(x-y)-f(x)) D_N(y)\,dy \right|}_{B} + \underbrace{\int_{-\pi}^{-\delta}}_{B'}$$

where δ will be chosen later.

$$|A| \le \frac{1}{2\pi}\int_{-\delta}^{\delta} \underbrace{|f(x-y)-f(x)|}_{\le [f]_\alpha |y|^\alpha}\underbrace{|D_N(y)|}_{\le C\frac{1}{|y|}}dy$$

$$\le \frac{1}{2\pi}C[f]_\alpha \int_{-\delta}^{\delta}|y|^{\alpha-1}dy$$

$0<\alpha\le 1$

$$= \frac{1}{2\pi}C[f]_\alpha\, 2\left[\frac{1}{\alpha}y^\alpha\right]_0^\delta$$

$$= \frac{C}{\pi}[f]_\alpha\, \delta^\alpha/\alpha$$

$$B = \frac{1}{2\pi}\int_\delta^\pi (f(x-y)-f(x))D_N(y)\,dy$$

$$= \int_\delta^\pi \frac{f(x-y)-f(x)}{2\pi\sin(y/2)}\sin((N+\tfrac{1}{2})y)\,dy$$

$$= -\int_\delta^\pi h_x(y)\sin\left((N+\tfrac{1}{2})\left(y+\frac{\pi}{N+\frac{1}{2}}\right)\right)dy$$

where $\quad h_x(y) = \dfrac{f(x-y)-f(x)}{2\pi\sin(y/2)}$

Thus

$$2B = B + B$$

$$= \int_\delta^\pi h_x(y)\sin((N+\tfrac{1}{2})y)\,dy - \int_\delta^\pi h_x(y)\sin\left((N+\tfrac{1}{2})\left(y+\frac{\pi}{N+\frac{1}{2}}\right)\right)dy$$

$$= \int_\delta^\pi h_x(y)\sin((N+\tfrac{1}{2})y)\,dy - \int_{\delta+\frac{\pi}{N+\frac{1}{2}}}^{\pi+\frac{\pi}{N+\frac{1}{2}}} h_x\left(y-\frac{\pi}{N+\frac{1}{2}}\right)\sin((N+\tfrac{1}{2})y)\,dy$$

$$= \int_{\delta}^{\pi} \left(h_x(y) - h_x\left(y - \frac{\pi}{N+\frac{1}{2}}\right)\right) \sin\left(\left(N+\tfrac{1}{2}\right) y\right) dy$$

$$- \int_{\pi}^{\pi + \frac{\pi}{N+\frac{1}{2}}} h_x\left(y - \frac{\pi}{N+\frac{1}{2}}\right) \sin\left(\left(N+\tfrac{1}{2}\right) y\right) dy$$

$$+ \int_{\delta}^{\delta + \frac{\pi}{N+\frac{1}{2}}} h_x\left(y - \frac{\pi}{N+\frac{1}{2}}\right) \sin\left(\left(N+\tfrac{1}{2}\right) y\right) dy$$

Since f is α-Hölder continuous it is in particular continuous and thus bounded on the circle, say by $M > 0$.

$$|h_x(y)| = \left| \frac{f(x-y) - f(x)}{2\pi \sin(y/2)} \right|$$

$$\leq \frac{2M}{2\pi \left|\frac{y}{\pi}\right|}$$

$$\leq \frac{M}{\delta} \quad \text{if} \quad \delta \leq y \leq \pi$$

$|\sin(x)| \geq \frac{|x|}{\pi/2}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

so $|\sin(\frac{x}{2})| \geq \frac{|x|}{\pi}$ on $[-\pi, \pi]$

If $\delta \leq y \leq \pi$ and $\delta > 2\tau = 2 \cdot \frac{\pi}{N+\frac{1}{2}}$

$$|h_x(y) - h_x(y - \tau)|$$

$$= \left| \frac{f(x-y) - f(x)}{2\pi \sin(y/2)} - \frac{f(x-y+\tau) - f(x)}{2\pi \sin((y-\tau)/2)} \right|$$

$$= \left| \frac{f(x-y)-f(x)}{2\pi \sin(y/2)} - \frac{f(x-y)-f(x)}{2\pi \sin((y-\tau)/2)} - \frac{f(x-y+\tau)-f(x-y)}{2\pi \sin((y-\tau)/2)} \right|$$

$$\leq |f(x-y)-f(x)| \frac{|\sin((y-\tau)/2) - \sin(y/2)|}{2\pi |\sin(y/2) \sin((y-\tau)/2)|}$$

$$+ |f(x-y+\tau) - f(x-y)| \frac{1}{2\pi |\sin(\frac{1}{2}(y-\tau))|}$$

$$\leq 2M \frac{\cancel{c'} \frac{1}{2}|\tau|}{2\pi \frac{|y|}{\pi} \frac{|y-\tau|}{\pi}} + [f]_\alpha |\tau|^\alpha \frac{1}{2\pi \frac{|y-\tau|}{\pi}}$$

$|\sin(x) - \sin(y)| \leq |x-y|$

~~For (ex) (by) looking at~~

~~$F(x,y) = (\sin(x) - \sin(y)) - (x-y)$~~

~~And optimize on $B_{10^3}(0,0)$~~

Mean value theorem

$|\sin(x) - \sin(y)| = |\cos(c)||x-y|$

$$= \frac{\pi M \cancel{c'}}{2} \frac{|\tau|}{\delta \frac{\delta}{2}} + [f]_\alpha |\tau|^\alpha \frac{1}{2 \cdot \frac{\delta}{2}}$$

$$= \pi M \cancel{c'} \frac{|\tau|}{\delta^2} + [f]_\alpha \frac{\tau^\alpha}{\delta}$$

Then using $\tau = \frac{\pi}{N+\frac{1}{2}} \leq 4N^{-1}$

$$|B| \leq \frac{1}{2}\left(\pi \left(\pi M \cancel{c'} \frac{4N^{-1}}{\delta^2} + [f]_\alpha \frac{4^\alpha N^{-\alpha}}{\delta} \right) + \frac{M}{\delta} 4N^{-1} + \frac{M}{\delta} 4N^{-1} \right)$$

provided $\delta > 2\tau = 2\frac{\pi}{N+\frac{1}{2}}$. Similarly bound B'.

Choose $\delta = N^{-\alpha/3}$. Get

$$
\begin{aligned}
&|S_N(f)(x) - f(x)| \\
&\le \frac{C}{\pi} \frac{[f]_\alpha}{\alpha} N^{-\alpha^2/3} + 4\pi^2 M C' N^{-1+\frac{2\alpha}{3}} + \frac{4^\alpha}{\pi} N^{-2\alpha/3} \\
&\quad + 4MN^{-1+\frac{\alpha}{3}} + 4MN^{-1+\frac{\alpha}{3}}
\end{aligned}
$$

$\longrightarrow 0$ as $N \to \infty$ uniformly in x. $\blacksquare$