Fourier Series Meets Linear Algebra
Part 4: The Dirichlet kernel

P1) Simplify

$$\sin(x) + \sin(3x) + \ldots + \sin((2n-1)x)$$

[Hint: Multiply this by a trig function and simplify in a similar manner as we did with the Dirchlet kernel.]

P2) Consider the sequence of functions

$$S_n(x) = \frac{nx}{1 + n^2x^2}$$

a) Show that as $n \to \infty$ we have $S_n(x) \to 0$ for every x.

b) Show that for every $n > 0$ there exist numbers (possibly depending on n) x_+ and x_- such that $S_n(x_+) = \frac{1}{2}$ and $S_n(x_-) = -\frac{1}{2}$. This is an example of the so-called Gibbs phenomenon.

c) Look up online information about the Gibbs phenomenon for Fourier series.

P3) Let $B_n(t)$ be defined in $-\pi < x < \pi$ to be equal to $\frac{n}{2}$ in the interval $-\frac{1}{n} < x < \frac{1}{n}$ and zero elsewhere in the interval, and to be periodic with period 2π.

a) If f is continuously differentiable and of period 2π show

$$\lim_{n\to\infty} (f * B_n)(x) = f(x)$$

b) Does the Gibbs phenomenon occur here?

c) Discuss the difference between B_n and the Dirichlet kernel D_n. You are highly encouraged to formulate hypothesis on why they sometimes have similar behavior and why they sometimes have different behavior.