Fourier Series Meets Linear Algebra
Part 1: Inner products and norms

P1) Show from the definition of an inner product that

$$\langle \alpha_1 f_1 + \beta_1 g_1, \alpha_2 f_2 + \beta_2 g_2 \rangle = \alpha_1 \overline{\alpha_2} \langle f_1, f_2 \rangle + \alpha_1 \overline{\beta_2} \langle f_1, g_2 \rangle + \beta_1 \overline{\alpha_2} \langle g_1, f_2 \rangle + \beta_1 \overline{\beta_2} \langle g_1, g_2 \rangle$$

P2) Find the inner product $\langle f, g \rangle$ where f, g are functions on $[-\pi, \pi]$ such that f is odd and g is even.

P3) Prove the parallelogram equality

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

P4) a) Complete the proof of the Cauchy-Schwarz inequality.

b) Prove in detail that if equality holds in the Cauchy-Schwarz inequality one of the vectors is a scalar multiple of the other.